

# Mathematics for Computer Science: Homework 6

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## 1 Exercise 6.10.22

We are given  $n + 1$  numbers from the set  $\{1, 2, \dots, 2n\}$ . Prove that there are two numbers among them such that one divides the other.

**Answer:**

**Proof (Pigeonhole Principle)** Denote the largest odd factor of a positive integer  $x$  as  $f(x) = \max\{2i + 1 : 2i + 1 \mid x\}$ . There are  $n$  possible  $f(x)$  values, namely all odd positive integers between 1 and  $2n$ :  $1, 3, \dots, 2n - 1$ . Thus there exists at least two integers  $x, y$  which must have the same largest odd factor  $f(x) = f(y)$  among  $n + 1$  selected integers by Pigeonhole Principle. If suppose  $x < y$  without loss of generality, then  $x \mid y$  with the quotient being a 2-power. ■

**Proof (Induction Principle)** It's obviously true when  $n = 1$ . Suppose it's true for  $n$ . Now look at the case of  $n + 2$  numbers between 1 and  $2n + 2$ . If  $n + 1$  of them are between 1 and  $2n$ , we're done by the induction hypothesis. Hence the remaining case is  $n$  of them between 1 and  $2n$ , and  $2n + 1, 2n + 2$  are in the set of these  $n + 2$  numbers.

Suppose that there doesn't exist two numbers in these  $n + 2$  numbers such that one divides the other. Denote the set of these  $n + 2$  numbers as  $S$ . Because  $n + 1$  divides  $2n + 2$ ,  $n + 1$  is not in  $S$ . Now we replace  $2n + 2$  with  $n + 1$  in  $S$  and get a new set  $S'$ . There exists  $n + 1$  numbers in  $S'$  which are less than or equal to  $2n$ . By the induction hypothesis, we can find  $x, y \in S'$  such that  $x < y$  and  $x \mid y$ . Because of the supposed property of  $S$ ,  $y$  must be  $n + 1$ . So  $x \mid n + 1 \mid 2n + 2$ . The contradiction is derived. Thus, there must exist two numbers in  $S$  such that one divides the other. ■

## 2 Exercise 6.10.23

What is the number of positive integers not larger than 210 and not divisible by 2, 3 or 7?

**Answer:**

Let  $T = \{1, 2, \dots, 210\}$ ,  $A = \{x \in T : 2 \mid x\}$ ,  $B = \{x \in T : 3 \mid x\}$ ,  $C = \{x \in T : 7 \mid x\}$ . The solution by InclusionExclusion Formula:

$$\begin{aligned} |\overline{A \cup B \cup C}| &= |T| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \\ &= 210 - \left\lfloor \frac{210}{2} \right\rfloor - \left\lfloor \frac{210}{3} \right\rfloor - \left\lfloor \frac{210}{7} \right\rfloor + \left\lfloor \frac{210}{2 \times 3} \right\rfloor + \left\lfloor \frac{210}{2 \times 7} \right\rfloor + \left\lfloor \frac{210}{3 \times 7} \right\rfloor - \left\lfloor \frac{210}{2 \times 3 \times 7} \right\rfloor \\ &= 210 - 105 - 70 - 30 + 35 + 15 + 10 - 5 \\ &= 60 \end{aligned}$$

Another solution based on the independencies between  $A$ ,  $B$  and  $C$ :

$$\begin{aligned} |\overline{A \cup B \cup C}| &= |\overline{A} \cap \overline{B} \cap \overline{C}| \\ &= 210 \times \frac{1}{2} \times \frac{2}{3} \times \frac{6}{7} = 210 \times \frac{2}{7} = 60 \end{aligned}$$

### 3 Special Problem 1

Let  $X_1, X_2, \dots, X_n$  be independent Poisson trials such that  $\Pr\{X_i = 1\} = p_i$ . Let  $X = \sum_{1 \leq i \leq n} X_i$  and  $\mu = E(X)$ . In class we derived one version of the Chernoff Bounds regarding the probability that  $X > (1 + \delta)\mu$ . Here you are asked to prove the following bounds in a similar way:

(a) For  $0 < \delta < 1$ ,

$$\Pr\{X \leq (1 - \delta)\mu\} \leq \left( \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^\mu.$$

(b) Assume that  $p_i = 1/2$  for all  $i$ . Prove the stronger bound that

$$\Pr\{|X - \frac{n}{2}| > a\} \leq 2e^{-\frac{a^2}{n}}.$$

(Hint: First show that  $e^t + 1 \leq 2e^{t/2+t^2/8}$  for all  $t > 0$ .)

Answer:

(a) **Lemma 3.1 (Markov's Inequality)** If  $X$  is any non-negative random variable and  $a > 0$ ,

$$\Pr\{X \geq a\} \leq \frac{E(X)}{a}$$

**Proof**

$$E(X) = \sum_{x=0}^{\infty} x \Pr\{X = x\} \geq \sum_{x=a}^{\infty} x \Pr\{X = x\} \geq \sum_{x=a}^{\infty} a \Pr\{X = x\} = a \Pr\{X \geq a\} \quad \blacksquare$$

$$\begin{aligned} \Pr\{X \leq (1 - \delta)\mu\} &= \Pr\{e^{tX} \geq e^{t(1-\delta)\mu}\} && (\text{Let } t = \ln(1 - \delta) < 0) \\ &\leq \frac{E(e^{tX})}{\exp(t(1 - \delta)\mu)} && (\text{by Markov's Inequality}) \\ &= \frac{\prod_i E(e^{tX_i})}{\exp(t(1 - \delta)\mu)} && (X = \sum_i X_i) \\ &= \frac{\prod_i (e^t \Pr\{X_i = 1\} + \Pr\{X_i = 0\})}{\exp(t(1 - \delta)\mu)} && (E(e^{tX_i}) = e^t \Pr\{X_i = 1\} + e^0 \Pr\{X_i = 0\}) \\ &= \frac{\prod_i (1 + (e^t - 1)p_i)}{\exp(t(1 - \delta)\mu)} && (\Pr\{X_i = 0\} = 1 - \Pr\{X_i = 1\} = 1 - p_i) \\ &\leq \frac{\prod_i \exp((e^t - 1)p_i)}{\exp(t(1 - \delta)\mu)} && (1 + x \leq e^x \text{ where } x \leq 0) \\ &= \frac{\exp((e^t - 1) \sum_i p_i)}{\exp(t(1 - \delta)\mu)} && (X = \sum_i X_i) \\ &= \frac{\exp((e^t - 1)\mu)}{\exp(t(1 - \delta)\mu)} && (\sum_i p_i = E(X) = \mu) \\ &= \left( \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^\mu && (e^t - 1 = e^{\ln(1-\delta)} - 1 = -\delta) \end{aligned}$$

(b)

$$E(X) = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n (1 \cdot Pr\{X_i = 1\} + 0 \cdot Pr\{X_i = 0\}) = \frac{n}{2}$$

$$\begin{aligned} Pr\{|X - \frac{n}{2}| > a\} &= Pr\{X > \frac{n}{2} + a\} + Pr\{X < \frac{n}{2} - a\} \\ &= Pr\{X > (1 + \delta)\mu\} + Pr\{X < (1 - \delta)\mu\} \quad (\text{Let } \delta = \frac{2a}{n}, \mu = E(X) = \frac{n}{2}) \\ &= Pr\{e^{tX} \geq e^{t(1+\delta)\mu}\} + e^{-\mu\delta^2/2} \\ &\quad (\text{Let } t > 0, \text{ and apply Special Problem 2(b), Homework 5}) \\ &\leq \frac{E(\exp(tX))}{\exp(t(1 + \delta)\mu)} + e^{-\mu\delta^2/2} \quad (\text{by Markov's Inequality}) \\ &= \frac{\prod_i (e^t/2 + 1/2)}{\exp(t(1 + \delta)\mu)} + e^{-\mu\delta^2/2} \quad (\text{Substitute } p_i = \frac{1}{2}) \\ &\leq \frac{\prod_i (2 \exp(t/2 + t^2/8)/2)}{\exp(t(1 + \delta)\mu)} + e^{-\mu\delta^2/2} \quad (\text{Lemma 3.2}) \\ &= \exp\left(n\left(\frac{t}{2} + \frac{t^2}{8}\right) - t\left(1 + \frac{2a}{n}\right)\frac{n}{2}\right) + \exp\left(-\frac{1}{2}\frac{n}{2}\left(\frac{2a}{n}\right)^2\right) \\ &= \exp\left(-\frac{3a^2}{2n}\right) + \exp\left(-\frac{a^2}{n}\right) \quad (\text{Let } t = \delta) \\ &\leq 2 \exp\left(-\frac{a^2}{n}\right) \end{aligned}$$

**Lemma 3.2** For  $t > 0$ ,

$$e^t + 1 \leq 2e^{t/2+t^2/8}$$

**Proof**

$$\begin{aligned} e^t + 1 &\leq 2e^{t/2+t^2/8} \\ \iff \frac{e^z + e^{-z}}{2} &\leq \exp\frac{z^2}{2} \quad (\text{Let } z = \frac{t}{2}) \\ \iff \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} &\leq \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n \cdot n!} \end{aligned}$$

Obviously,

$$\prod_{i=n+1}^{2n} i \geq 2^n$$

holds. So we are done. ■

## 4 Special Problem 2

Let  $T_n$  denote the set of all possible ways to parenthesis the arithmetic expression  $x_1 + x_2 + \cdots + x_n$ . In class we used generating function techniques to show that  $|T_n| = \binom{2n-2}{n-1}/n$ . We now derive this result in a

different way. In Homework Set 3, Special Problem 3, it was shown that, if  $P_n$  is the set all no-debt *paying patterns* for  $2n$  people, then  $|P_n| = \binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n}/(n+1)$ .

- (a) Give an explicit one-to-one and onto mapping  $f : T_n \rightarrow P_{n-1}$ .
- (b) Describe an algorithm that, given as input an expression  $t \in T_n$ , outputs  $f(t) \in P_{n-1}$ . Your algorithm needs to run in time linear in  $n$ .

**Answer:**

We only reserve the left bracket symbol '(' and the plus symbol '+' in the string of the arithmetic expression parenthesised with  $n+1$  variables. For example, "((+(+)" is the string what we interested in from  $((x_1 + (x_2 + x_3)) + x_4)$ . There are  $n$  '('s and  $n$  '+'s such that the number of '+' occurrence is not less than the number of '(' occurrence anytime if we read the string from left to right. The sequence satisfied the definition of a *valid* paying pattern if we replace '(' with '+1' and '+' with '-1'. Thus, a linear algorithm  $F$  for problem (b) is obviously obtained: reserve '(' and '+' in the expression and then replace '(' with '+1' and '+' with '-1'.

In the other hand, if given a *valid* string which consists of  $n$  '('s and  $n$  '+'s, a valid parenthesised expression can be also rebuild by the following algorithm  $G$ .

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**Algorithm 1** Map  $P_{n+1}$  to  $T_n$

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**Require:** Given a sequence which consists of  $n$  '+1' and  $n$  '-1'.

**Ensure:** Return the string of the parenthesised expression with  $n+1$  variables.

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1: function G(s)
2:   if s is empty then
3:     return x
4:   end if
5:   pos ← (the first position that the partial sum of the sequence s is equal to 0).
6:   return '(' + (G(s2...pos-1)) + '+' + (G(spos+1...2n)) whose the indices of x are offsetted + ')'
7: end function

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Therefore, we obtain the one-to-one correspondence between  $T_{n+1}$  and  $P_n$ .

## 5 Special Problem 3

Solve each of the following recurrence relations:

- (a)  $a_0 = 1, a_1 = 2, a_n = 4a_{n-1} - 4a_{n-2} + 3n + 1$  for all  $n \geq 2$ .
- (b)  $a_0 = 1, a_n = \frac{a_{n-1}}{3+4a_{n-1}}$  for  $n \geq 1$ .

**Answer:**

- (a) Denote the generating function by  $G(x) = \sum_{n=0}^{\infty} a_n x^n$ . The recurrence relations are held when  $n \geq 2$ . So we have

$$G(x) - a_0 - a_1 x = 4x(G(x) - a_0) - 4x^2 G(x) + 3x \left( \frac{x^2}{1-x} \right)' + \frac{x^2}{1-x}$$

We derive that

$$\begin{aligned}
 G(x) &= \frac{-6x^3 + 12x^2 - 4x + 1}{(2x^2 - 3x + 1)^2} \\
 &= \frac{10}{1-x} + \frac{3}{(1-x)^2} + \frac{-17}{1-2x} + \frac{5}{(1-2x)^2} \\
 &= \frac{10}{1-x} + \left(\frac{3}{1-x}\right)' + \frac{-17}{1-2x} + \left(\frac{5/2}{1-2x}\right)' \\
 &= \sum_{n=0}^{\infty} [10 + 3 \cdot (n+1) - 17 \cdot 2^n + 5 \cdot 2^n(n+1)] x^n \\
 &= \sum_{n=0}^{\infty} [13 + 3n - 12 \cdot 2^n + 5n2^n] x^n
 \end{aligned}$$

So

$$a_n = 13 + 3n - 12 \cdot 2^n + 5n2^n$$

(b) Let  $b_n = 1/a_n$ . Substituting  $a_n$  by  $1/b_n$ ,

$$a_n = \frac{a_{n-1}}{3 + 4a_{n-1}} \implies 1/b_n = \frac{1/b_{n-1}}{3 + 4/b_{n-1}}$$

We derive that  $b_n = 3b_{n-1} + 4$ . Denote the generating function by  $G(x) = \sum_{n=0}^{\infty} b_n x^n$ . The recurrence relations are held when  $n \geq 1$ . So we have

$$G(x) - b_0 = 3xG(x) + \frac{4x}{1-x}$$

We derive that

$$\begin{aligned}
 G(x) &= \frac{1 + 3x}{(1-x)(1-3x)} \\
 &= \frac{3}{1-3x} - \frac{2}{1-x} \\
 &= \sum_{n=0}^{\infty} (3 \cdot 3^n - 2) x^n
 \end{aligned}$$

So

$$a_n = \frac{1}{3^{n+1} - 2}$$

## 6 Special Problem 4

Finish the derivation of the formula for the number of ways to tile a  $3 \times n$  board by dominoes.

**Answer:**

Let  $G$  be the generating function of the number of ways to tile a  $3 \times n$  board by  $2 \times 1$  dominoes. Let  $U$  be the generating function of the number of ways to tile a  $3 \times n + 1 \times 2$  (down) board by  $2 \times 1$  dominoes.

$$U = \overline{\square \dots}$$

Let  $V$  be the generating function of the number of ways to tile a  $3 \times n + 1 \times 2$  (up) board by  $2 \times 1$  dominoes.

$$V = \overline{\square \cdot \cdot}$$

Define the operator “ $\cdot$ ” as to concatenate the two boards satisfying the distributive law to “ $+$ ” in the generating functions. According to the state transitions, we figure the transition equations as following.

$$G = \overline{\square} + \overline{\square} \cdot U + \overline{\square} \cdot V + \overline{\square} \cdot G$$

$$U = \overline{\square} \cdot U + \overline{\square} \cdot G$$

$$V = \overline{\square} \cdot V + \overline{\square} \cdot G$$

Replacing the domino shapes with the power of  $x$ , solve the equations of generating functions:

$$G(x) = 1 + x^2U + x^2V + x^3G$$

$$U(x) = x^3U + xG$$

$$V(x) = x^3V + xG$$

We have

$$\begin{aligned} G(x) &= \frac{1 - x^3}{1 - 4x^3 + x^6} \\ &= \frac{(-3 + \sqrt{3})/6}{x^3 - 2 + \sqrt{3}} + \frac{(-3 - \sqrt{3})/6}{x^3 - 2 - \sqrt{3}} \\ &= \frac{(3 + \sqrt{3})/6}{1 - (2 + \sqrt{3})x^3} + \frac{(3 - \sqrt{3})/6}{1 - (2 - \sqrt{3})x^3} \\ &= \sum_{n=0}^{\infty} \left( \frac{3 - \sqrt{3}}{6} (2 + \sqrt{3})^n + \frac{3 + \sqrt{3}}{6} (2 - \sqrt{3})^n \right) x^{3n} \end{aligned}$$

## 7 Special Problem 5

A  $2 \times n$  matrix  $A = (a_{i,j})$  is called regular if the following conditions are satisfied:

- (i) the  $2n$  entries of  $A$  are exactly the integers  $1, 2, \dots, 2n - 1, 2n$ , with each integer appearing exactly once;
- (ii) For each  $i = 1, 2$ , we have  $a_{i,1} < a_{i,2} < \dots < a_{i,n}$ ;
- (iii) For each  $1 \leq j \leq n$ , we have  $a_{1,j} < a_{2,j}$ .

*Question:* Derive an explicit expression for the number of such regular  $2 \times n$  matrices.

**Answer:**

For a valid configuration, the largest number  $2n$  must occur in  $a_{2,n}$ . After  $2n$  removed, the second largest number  $2n - 1$  must occur in  $a_{1,n}$  or  $a_{2,n-1}$ . After  $2n - 1$  removed, the third largest number  $2n - 2$  must occur in the last nonempty cell of the first row or the second row, and in addition, the position of the last nonempty cell of the second row is not larger than the first row. As the process above, we repeat to remove the largest number in the current configuration. We can conclude that if there are  $m$  remaining numbers in

the current configuration, the largest number  $m$  must occur in the last nonempty cell of the first row or the second row, and in addition, the position of the last nonempty cell of the second row is not larger than the first row.

This conclusion can be also obtained in the opposite direction. Namely, each time, we add the smallest number which doesn't occur in the current configuration into the first empty cell of the first row or the second row, and keep the filled cells in the first row are more than in the second row. So far, we have builded the one-to-one correspondence between a valid configuration and the method we put the number from small to large which is described above.

If we write down the mark '+1' if we fill the number in the first row and the mark '-1' if we fill the number in the second row and list the marks as a sequence in order we add into the grid, this mark sequence is equivalent to the sequence in the paying pattern model in Homework Set 3, Special Problem 3. Therefore, the answer is

$$\frac{1}{n+1} \binom{2n}{n}$$