Mathematics for Computer Science: Homework 4

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Contents

1 Exercise 4.3.7

How many subsets does the set $\{1, 2, \cdots, n\}$ have that contain no three consecutive integers? Find a recurrence.

Answer:

Denote F_i as the number of the subsets containing no three consecutive integers. Initial values is $F_0 = 1, F_1 = 2, F_2 = 4$. The recurrence is

$$F_n = F_{n-1} + F_{n-2} + F_{n-3}$$

If we don't choose the n-th element, there is F_{n-1} ways to choose the first $n-1$ elements. Otherwise if we choose the n-th element and don't choose the $(n-1)$-th element, there is F_{n-2} ways to choose the first $n-2$ elements. Otherwise if we choose the $(n-1)$th, n-th element and don't choose the $(n-2)$-th element, there is F_{n-3} ways to choose the first $n-3$ elements.

2 Exercise 4.3.9

Prove the following identities:

(a) $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$;

(b) $F_{n+1}^2 - F_n^2 = F_{n-1}F_{n+2}$;

(c) $\binom{n}{0}F_0 + \binom{n}{1}F_1 + \binom{n}{2}F_2 + \cdots + \binom{n}{n}F_n = F_{2n}$;

(d) $\binom{n}{0}F_1 + \binom{n}{1}F_2 + \binom{n}{2}F_3 + \cdots + \binom{n}{n}F_{n+1} = F_{2n+1}$.

Answer:

(a) For $n = 1$, $F_2 = F_3 - 1$. Suppose that it holds for $n - 1$.

$$\begin{aligned}
F_2 + F_4 + F_6 + \cdots + F_{2n} &= F_2 + F_4 + F_6 + \cdots + F_{2(n-1)} + F_{2n} \\
&= F_{2(n-1)+1} - 1 + F_{2n} \\
&= F_{2n-1} + F_{2n} - 1 \\
&= F_{2n+1} - 1
\end{aligned}$$

(b) $F_{n+1}^2 - F_n^2 = (F_{n+1} + F_n)(F_{n+1} - F_n) = F_{n+2}F_{n-1}$

(c), (d) For $n = 0$, $\binom{0}{0}F_0 = F_0$ and $\binom{0}{0}F_1 = F_1$. Suppose that they hold for $n - 1$.

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} F_k &= \sum_{k=0}^{n} \left[\binom{n-1}{k-1} F_k + \binom{n-1}{k} F_k \right] \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} F_{k+1} + \sum_{k=0}^{n-1} \binom{n-1}{k} F_k \\
&= F_{2(n-1)+1} + F_{2(n-1)} \\
&= F_{2n}
\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}F_{k+1} &= \sum_{k=0}^{n}\left[\binom{n-1}{k-1}F_{k+1}+\binom{n-1}{k}F_{k+1}\right]\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}F_{k+2}+\sum_{k=0}^{n-1}\binom{n-1}{k}F_{k+1}\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}(F_k+F_{k+1})+\sum_{k=0}^{n-1}\binom{n-1}{k}F_{k+1}\\ &= F_{2(n-1)}+F_{2(n-1)+1}+F_{2(n-1)+1}\\ &= F_{2n}+F_{2n-1}\\ &= F_{2n+1}\end{aligned}$$

3 Exercise 4.3.16

(a) Prove that every positive integer can be written as the sum of different Fibonacci numbers.

(b) Prove even more: every positive integer can be written as the sum of different Fibonacci numbers, so that no two consecutive Fibonacci numbers are used.

(c) Show by an example that the representation in (a) is not unique, but also prove that the more restrictive representation in (b) is.

Answer:

(a) If (b) has been proved, (a) is obvious.

(b) This problem is as known as Zeckendorf's theorem. Its proof can be divided into two parts.

Proof (Existence: every positive integer n has a Zeckendorf representation) Denote $Z(n)$ as the Zeckendorf representation of n.

For $n = 1, 2$, it's true. Suppose that each positive integer less than n has a Zeckendorf representation. If n is a Fibonacci number, we get the representation. Otherwise, there exists i such that $F_i < n < F_{i+1}$. So $n - F_i < Fi + 1 - F_i = F_{i-1}$.

By induction hypothesis, all numbers of the Zeckendorf representation of $n - F_i$ are less than F_i. Then n can be represented as $Z(n - F_i) + F_i$ where the numbers in $Z(n - F_i)$ are different with F_i. ■

(c) **Lemma 3.1** *The sum of any non-empty set S of distinct, non-consecutive Fibonacci numbers whose largest member is F_i is strictly less than the next largest Fibonacci number F_{i+1}.*

Proof It's obvious when there is only one element in the set. Suppose that the lemma holds for $S \setminus \{F_i\}$ whose maximum element is F_j $(j < i - 1)$.

$$Sum(S) = Sum(S \setminus \{F_i\}) + F_i < F_{j+1} + F_i \le F_{i-1} + F_i = F_{i+1}$$

Proof (Uniqueness: no positive integer n has two different Zeckendorf representations) Take S and T are two sets of the distinct non-consecutive Fibonacci numbers which have the same sum. Let $S' = S \setminus T$, $T' = T \setminus S$.

Suppose S' and T' are both non-empty. Let the maximum element of S' and T' be F_i and F_j respectively. Without loss of generality, suppose $F_i < F_j$. By applying Lemma, $Sum(S') < F_{i+1} \leq F_j < Sum(T')$. Contradiction occurs.

If S' is empty and T' is non-empty, then the sum of S and T are different. Contradiction occurs. So the uniqueness holds. ■

Example 3.2 12 *can be written in* $12 = 8+3+1 = 5+3+2+1+1$. *The representation is not unique. But its Zeckendorf representation, namely* $12 = 8+3+1$ *is unique.* ■

4 Exercise 5.4.2

Choose an integer uniformly from the set $\{1, 2, 3, \cdots, 30\}$. Let A be the event that it is divisible by 2; let B be the event that it is divisible by 3; let C be the event that it is divisible by 7.

(a) Determine the probabilities of A, B, and C.

(b) Which of the pairs (A, B), (B, C), and (A, C) are independent?

Answer:

(a)

$$\Pr(A) = \frac{\lfloor \frac{30}{2} \rfloor}{30} = \frac{15}{30} = \frac{1}{2}$$
$$\Pr(B) = \frac{\lfloor \frac{30}{3} \rfloor}{30} = \frac{10}{30} = \frac{1}{3}$$
$$\Pr(C) = \frac{\lfloor \frac{30}{7} \rfloor}{30} = \frac{4}{30} = \frac{2}{15}$$

(b)

$$\Pr(A \cap B) = \frac{\lfloor \frac{30}{2\cdot 3} \rfloor}{30} = \frac{5}{30} = \frac{1}{6} = \Pr(A)\Pr(B)$$
$$\Pr(B \cap C) = \frac{\lfloor \frac{30}{3\cdot 7} \rfloor}{30} = \frac{1}{30} = \frac{1}{30} \neq \Pr(B)\Pr(C)$$
$$\Pr(A \cap C) = \frac{\lfloor \frac{30}{2\cdot 7} \rfloor}{30} = \frac{2}{30} = \frac{1}{15} = \Pr(A)\Pr(C)$$

So (A, B), (A, C) are independent.

5 Exercise 5.4.5

We flip a coin n times ($n \geq 1$). For which values of n are the following pairs of events independent?

(a) The first coin flip was heads; the number of all heads was even.

(b) The first coin flip was head; the number of all heads was more than the number of tails.

(c) The number of heads was even; the number of heads was more than the number of tails.

Answer:

$$
\begin{aligned}
\Pr(A) &= \Pr\{\text{the first coin flip was head}\} \\
&= \frac{1}{2} \\
\Pr(B) &= \Pr\{\text{all heads was even}\} \\
&= \begin{cases} \frac{1}{2^n}\sum_{i=0}^{k}\binom{2k}{2i} = \frac{1}{2} & (n = 2k) \\ \\ \frac{1}{2^n}\sum_{i=0}^{k}\binom{2k+1}{2i} = \frac{1}{2} & (n = 2k+1) \end{cases} \\
\Pr(C) &= \Pr\{\text{all heads was more than the number of tails}\} \\
&= \begin{cases} \frac{1}{2^n}\sum_{i=0}^{k-1}\binom{2k}{i} = \frac{1}{2} - \frac{\Gamma(1/2+k)}{2\sqrt{\pi}\Gamma(1+k)} < \frac{1}{2} & (n = 2k) \\ \\ \frac{1}{2^n}\sum_{i=0}^{k}\binom{2k+1}{i} = \frac{1}{2} & (n = 2k+1) \end{cases}
\end{aligned}
$$

(a) When $n > 1$, we have

$$
\Pr(A \cap B) = \begin{cases} \frac{1}{2}\frac{1}{2^{n-1}}\sum_{i=0}^{k-1}\binom{2k-1}{2i+1} = \frac{1}{4} & (n = 2k) \\ \\ \frac{1}{2}\frac{1}{2^{n-1}}\sum_{i=0}^{k-1}\binom{2k}{2i+1} = \frac{1}{4} & (n = 2k+1) \end{cases} = \Pr(A)\Pr(B)
$$

When $n = 1$, $\Pr(A \cap B) = 0$. The pair of A and C is independent while $n > 1$.

(b)

$$
\Pr(A \cap C) = \begin{cases} \frac{1}{2}\frac{1}{2^{n-1}}\sum_{i=0}^{k-1}\binom{2k-1}{i} = \frac{1}{4} > \Pr(A)\Pr(C) & (n = 2k) \\ \\ \frac{1}{2}\frac{1}{2^{n-1}}\sum_{i=0}^{k}\binom{2k}{i} > \frac{1}{4} = \Pr(A)\Pr(C) & (n = 2k+1) \end{cases}
$$

The pair of A and C is not independent.

(c)

$$\Pr(B \cap C) = \begin{cases} \frac{1}{2^n} \sum_{i=k+1}^{2k} \binom{4k}{2i} \neq \Pr(B)\Pr(C) & (n = 4k) \\ \frac{1}{2^n} \sum_{i=k+1}^{2k} \binom{4k+1}{2i} < \frac{1}{4} = \Pr(B)\Pr(C) & (n = 4k+1) \\ \frac{1}{2^n} \sum_{i=k+1}^{2k+1} \binom{4k+2}{2i} = \frac{1}{4} > \Pr(B)\Pr(C) & (n = 4k+2) \\ \frac{1}{2^n} \sum_{i=k+1}^{2k+1} \binom{4k+3}{2i} > \frac{1}{4} = \Pr(B)\Pr(C) & (n = 4k+3) \end{cases}$$

The pair of B and C is not independent.

6 Special Problem 1

Show that, for even n, the bit-fixing routing strategy has worst-case running time greater than $2^{n/2-1}$.

Answer:

Let us consider the worst-case on the edge (i, j) where

$$i = (a_1, a_2, \cdots, a_k, \cdots, a_n), \quad j = (a_1, a_2, \cdots, 1 - a_k, \cdots, a_n)$$

Based on the bit-fixing routing strategy, a package v_x from x to $\sigma(x)$ via the edge (i, j) statisfies that the sequence $(a_1, a_2, \cdots, 1 - a_k)$ must be the prefix sequence of $\sigma(x)$ and the sequence $(a_k, a_{k+1}, \cdots, a_n)$ must be the suffix sequence of x. The first $k - 1$ elements of x are free variable. There are 2^{k-1} kinds of x. The last $n - k$ elements of $\sigma(x)$ are free variable. There are 2^{n-k} kinds of $\sigma(x)$. There are $\min\{2^{k-1}, 2^{n-k}\}$ packages via the edge (i, j). When $k = n/2$ where n is an even integer, $\min\{2^{k-1}, 2^{n-k}\}$ gets the maximum value $2^{n/2-1}$. So The worst-case running time is not less than $2^{n/2-1}$.

7 Special Problem 2

Let σ be a permutation so that for each node $j \in \{0, 1\}^n$. in the hypercube network, a packet v_j is to be routed to node $\sigma(j)$. For each node j, let $p_j = e_1 e_2 \cdots e_{l_j}$ be the path followed by packet v_j under the bit-fixing strategy. Now let i be any fixed node. Let S be the set of $j \neq i$ such that the paths p_j and p_i share at least one common edge. Show that the number of steps used in delivering packet v_i is no more than $l_i + |S|$. (That is, the extra delay for packet v_i is at most $|S|$.)

Answer:

For the fixed i, we project the path $p_i = e_1 e_2 \cdots e_{l_j}$ into x axis. The delay can be represented by a unit segment parallel to the time axis t. Each $j \in S$ has the common path with p_i. It can be shown in the figure below. We consider that p_i has been delayed by p_j at the point P_1 and has been delayed by p_j again at another point P_2. Obviously, if p_j touches p_i twice, p_j must be delayed by the certain path p_k at a certain point Q at the time between P_1 and P_2. Thus, we swap the path segment of p_j and p_k after the point Q (See figure below). The total delay time doesn't change after this kind of operation. However, through the several times of this kind of operation, we can guarantee that each $j \in S$, p_j touches the path p_i at most once. So v_i is delayed no more than $|S|$.

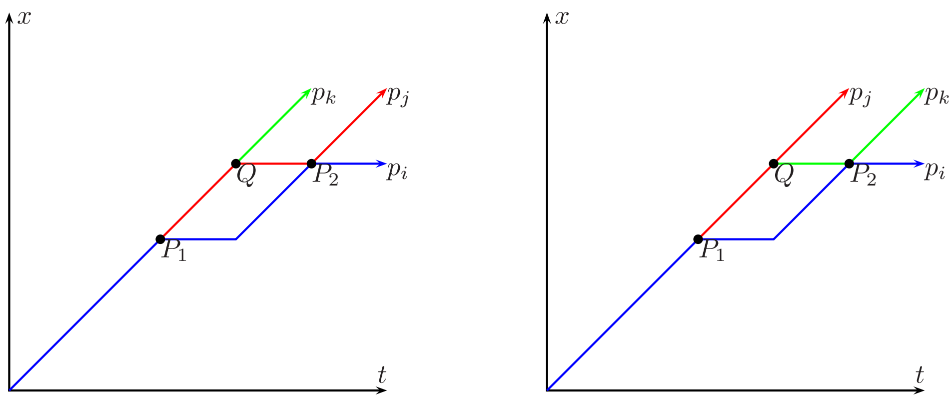

Figure 1: Operation: swap the path touching v_i. It don't change the total delay time.

8 Special Problem 3

Consider a *random walk* on a circle with nodes $v_0, v_1, \cdots, v_{n-1}$, and an edge between v_i and $v_{(i+1) \bmod n}$ for each of $i = 0, 1, 2, \cdots, n-1$. Starting initially at v_0 , in each step we move from the current node v_i randomly to either $v_{(i-1) \bmod n}$ or $v_{(i+1) \bmod n}$ with equal probability. After N steps, let p_N be the probability that all n nodes have been visited. Prove that $p_N \geq 1 - cn/N^{1/2}$ for some positive constant c.
Remark This implies that $p_N \to 1$ as $N \to \infty$. Thus, an infinite random walk on such a circle will eventually visit all the nodes.

Answer:

If we move consecutively n times in one direction, all nodes must be visited at least once. We denote the number of cases of moving N times without consecutively n times in one direction as F_N. Obviously,

$$p_N \geq 1 - \frac{F_N}{2^N}$$

Generalizing 4.3.7 in LPV, we have $F_N = F_{N-1} + F_{N-2} + \cdots + F_{N-n}$. F_N can be represented as $\sum_{i=1}^{n} c_i r_i^N$ where $\{r_i\}$ is the root of $x^n = x^{n-1} + x^{n-2} + \cdots + 1$.

Lemma 8.1 *Let r_m be the root of $x^n = x^{n-1} + x^{n-2} + \cdots + 1$ whose modulus is the maximum modulus among all roots. $|r_m|$ is strictly less than 2.*

Proof Obviously, $x \neq 0$. So we have

$$1 = \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n}$$

We suppose that $|x| > 1$ and get the modulus of the both side:

$$\begin{aligned}1 &\leq |\frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n}| \\ &= \frac{1}{|x|} + \frac{1}{|x^2|} + \cdots + \frac{1}{|x^n|} \\ &< \frac{1}{|x|} + \frac{1}{|x^2|} + \cdots + \frac{1}{|x^n|} + \frac{1}{|x^{n+1}|} + \cdots \\ &= \frac{1}{|x| - 1}\end{aligned}$$

After rearrangement, we have

$$|x| < 2$$

Thus if $|x| > 1$, then $|x| < 2$. So $|x| < 2$. ■

By the lemma above, we have

$$\begin{aligned}p_N &\geq 1 - \frac{F_N}{2^N} \\ &= 1 - \sum_{i=1}^{n} c_i \left(\frac{r_i}{2}\right)^N \\ &\geq 1 - nc_m \left(\frac{|r_m|}{2}\right)^N \\ &= 1 - nO((1 - o(1))^N) \\ &= 1 - nO(N^{-1/2})\end{aligned}$$

9 Special Problem 4

The probability space concept $\Omega = (U, p)$ can be generalized to the case when U is an infinite set $\{u_1, u_2, u_3, \cdots, u_n, \cdots\}$. As before, an *event* T is a subset of U , and $\Pr(T) = \sum_{u \in T} p(u)$. Consider a random walk on an n-node circle (as in the previous problem) such that it halts as soon as all the nodes have been visited. We are interested in the probability $q_{n,i}$ for the random walk to halt at node v_i (i.e. v_i is the very last node that the random walk visits). Clearly, for $n = 3$, we have $q_{3,1} = q_{3,2} = 1/2$.

(a) Specify a probability space $\Omega = (U, p)$ for this random walk. You may use an infinite U. Show that, according to your specification, $\sum_{u \in U} p(u) = 1$.

(b) Derive the values of $q_{4,i}$ for $i = 1, 2, 3$.

(c) Determine $q_{n,i}$ for $1 \leq i \leq n - 1$.

Answer:

(a) Let $U = \{0, 1, \cdots, n - 1\}$. Specify p_i as the probability we stay at node i after walking N steps. It's obvious that $\sum_{u \in U} p(u) = 1$.

(b) $q_{4,1} = q_{4,2} = q_{4,3} = \frac{1}{3}$ by applying (c).

(c) **Lemma 9.1** *Consider the line graph $S - A_1 - A_2 - \cdots - A_n - T$. Initially, we stay in A_1. Under the random walk model, the probability P_n of reaching S before T is $(n-1)/n$.*

Proof Obviously, $P_1 = \frac{1}{2}$. For $n > 1$,

$$\begin{array}{ccccccc} & & \text{Start} & & & & \\ & & \downarrow & & & & \\ S & \leftarrow & A_1 & \rightarrow & A_2 & \cdots & A_n \quad T \end{array}$$

Figure 2: Initially configuration

We can divide evaluation of P_n into these two parts using the law of total probability:

$$\begin{aligned} P_n &= \Pr\{\text{the first step is left}\} \cdot \Pr\{\text{reach } S \text{ before } T \mid \text{the first step is left}\} \\ &\quad + \Pr\{\text{the first step is right}\} \cdot \Pr\{\text{reach } S \text{ before } T \mid \text{the first step is right}\} \\ &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (P_{n-1} P_n) \end{aligned}$$

Evaluate Pr(reach S before T | the first step is right) as following. Because the first step is right, we stay in A_2 now. If we want to reach S, we must reach A_1 again before T with probability P_{n-1}. (Consider A_1 as S's role in the $(n-1)$-size Lemma). After going back to A_1, we need to reach S before T again with probability P_n.

$$\begin{array}{ccccccc} & & & \text{Now} & & & \\ & & & \downarrow & & & \\ S & A_1 & \leftarrow & A_2 & \cdots & A_n & T \end{array}$$

Figure 3: The configuration when the first step is right

Solve the recurrence equations by induction:

$$\begin{cases} P_1 = \dfrac{1}{2} \\ P_n = \dfrac{1}{2 - P_{n-1}} \end{cases} \Longrightarrow P_n = \frac{n}{n+1} \tag{9.1}$$

Thus, the probability of reaching T before S is $\frac{1}{n+1}$. ■

We can divide evaluation of $p_{n,i}$ into these two parts using the law of total probability:

$$\begin{aligned} p_{n,i} &= \Pr\{\text{reach } i+1 \text{ before } i-1\} \cdot \Pr\{\text{reach } i-1 \text{ before } i \mid \text{reach } i+1 \text{ before } i-1\} \\ &\quad + \Pr\{\text{reach } i-1 \text{ before } i+1\} \cdot \Pr\{\text{reach } i+1 \text{ before } i \mid \text{reach } i-1 \text{ before } i+1\} \\ &= \Pr\{\text{reach } i+1 \text{ before } i-1\} \cdot \frac{1}{n-1} + \Pr\{\text{reach } i-1 \text{ before } i+1\} \cdot \frac{1}{n-1} \\ &= [\Pr\{\text{reach } i+1 \text{ before } i-1) + \Pr\{\text{reach } i-1 \text{ before } i+1\}] \cdot \frac{1}{n-1} = 1 \cdot \frac{1}{n-1} \\ &= \frac{1}{n-1} \end{aligned}$$

$\Pr\{\text{reach } i-1 \text{ before } i \,|\, \text{reach } i+1 \text{ before } i-1\}$ is the probability of visiting all nodes at least once and ending up at i after reaching $i+1$ from 0 before $i-1$. After reaching $i+1$, we can reduce the circular graph into the line graph because we can not reach i before $i-1$. The edge from i to $i-1$ is disabled now. (See the figure below.) In the line graph, i, $i+1$, $i-1$ plays the roles of S, A_1, T in Lemma respectively, and there are $n-2$ nodes between i and $i-1$. So $\Pr\{\text{reach } i-1 \text{ before } i \,|\, \text{reach } i+1 \text{ before } i-1\} = \frac{1}{n-2+1}$. This case is the same as $\Pr\{\text{reach } i+1 \text{ before } i \,|\, \text{reach } i-1 \text{ before } i+1\}$.

$$\begin{array}{cccccccc} & \text{Now} & & & & & & \\ & \downarrow & & & & & & \\ i & i+1 & \cdots & n-1 & 0 & 1 & \cdots & i-1 \end{array}$$

Figure 4: The configuration after reaching $i+1$ from 0 before $i-1$

We don't need to know which values $\Pr\{\text{reach } i+1 \text{ before } i-1\}$ and $\Pr\{\text{reach } i-1 \text{ before } i+1\}$ are. Just use the property that their sum is 1.