Mathematics for Computer Science: Homework 2

Instructed by *Andrew C. Yao*

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Botao Hu J72 2007011292
`hupo001@gmail.com`

Contents

1 Exercise 2.5.1

What is the following sum?

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}$$

Experiment, conjecture the value, and then prove it by induction.

Answer:

We observe that

$$\begin{aligned}\frac{1}{1\cdot 2}&=\frac{1}{2}=1-\frac{1}{2}\\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}&=\frac{2}{3}=1-\frac{1}{3}\\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}&=\frac{3}{4}=1-\frac{1}{4}\\ &\cdots\end{aligned}$$

Proposition 1.1 *For each* $n\in\mathbb{N}^*\setminus\{1\}$*, we have*

$$\sum_{i=1}^{n-1}\frac{1}{i\cdot(i+1)}=1-\frac{1}{n}$$

Proof The proposition holds for $n=2$. Suppose it's true for $n-1$.

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{1}{i\cdot(i+1)}&=\sum_{i=1}^{n-2}\frac{1}{i\cdot(i+1)}+\frac{1}{(n-1)\cdot n}\\ &=1-\frac{1}{n-1}+\frac{1}{(n-1)\cdot n}\\ &=1-\frac{1}{n}\end{aligned}$$

The proposition is held for all $n\in\mathbb{N}^*\setminus\{1\}$ by the Induction Hypothesis. ■

2 Exercise 2.5.2

What is the following sum?

$$0\cdot\binom{n}{0}+1\cdot\binom{n}{1}+2\cdot\binom{n}{2}+\cdots+(n-1)\cdot\binom{n}{n-1}+n\cdot\binom{n}{n}$$

Experiment, conjecture the value, and then prove it. (Try to prove the result by induction and also by combinatorial arguments.)

Answer:

We observe that

$$\begin{aligned}
0\cdot\binom{1}{0}+1\cdot\binom{1}{1} &= 1 = 2^0\cdot 1\\
0\cdot\binom{2}{0}+1\cdot\binom{2}{1}+2\cdot\binom{2}{2} &= 4 = 2^1\cdot 2\\
0\cdot\binom{3}{0}+1\cdot\binom{3}{1}+2\cdot\binom{3}{2}+3\cdot\binom{3}{3} &= 12 = 2^2\cdot 3\\
0\cdot\binom{4}{0}+1\cdot\binom{4}{1}+2\cdot\binom{4}{2}+3\cdot\binom{4}{3}+4\cdot\binom{4}{4} &= 32 = 2^3\cdot 4\\
0\cdot\binom{5}{0}+1\cdot\binom{5}{1}+2\cdot\binom{5}{2}+3\cdot\binom{5}{3}+4\cdot\binom{5}{4}+5\cdot\binom{5}{5} &= 80 = 2^4\cdot 5
\end{aligned}$$

Proposition 2.1 *For each $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{n} i\cdot\binom{n}{i} = 2^{n-1}\cdot n$$

Proof The proposition holds for $n = 1$. Suppose it's true for $n - 1$.

$$\begin{aligned}
\sum_{i=0}^{n} i\cdot\binom{n}{i} &= \sum_{i=0}^{n-1} i\cdot\binom{n-1}{i} + \sum_{i=1}^{n} i\cdot\binom{n-1}{i-1}\\
&= \sum_{i=0}^{n-1} i\cdot\binom{n-1}{i} + \sum_{i=0}^{n-1}(i+1)\cdot\binom{n-1}{i}\\
&= 2\cdot\sum_{i=0}^{n-1} i\cdot\binom{n-1}{i} + \sum_{i=0}^{n-1}\binom{n-1}{i}\\
&= 2\cdot 2^{n-2}\cdot(n-1) + 2^{n-1}\\
&= 2^{n-1}\cdot n
\end{aligned}$$

Therefore, the proposition is held for all $n \in \mathbb{N}$ by the Induction Hypothesis. ■

The combinatorial argument corresponding to the formula is how many ways we color the n objects with 3 colors, one of which can be used exactly once and two remaining colors of which are unlimited.

We have $\binom{n}{i}$ choices to choose i of n objects to be colored with the special color and an another color. Thus, there are i ways to color the certain one in the chosen subsets with the special color, namely i ways to color the chosen subsets. Therefore, there are $\sum_{i=0}^{n} i\cdot\binom{n}{i}$ ways to statisfy the condition.

Here is another way to consider this model. We have n choices to choose a object for being colored with the special color. And there are 2^{n-1} ways to color the remaining $n-1$ objects with two colors. So the number of ways is $n\cdot 2^{n-1}$. Hence, the proposition is held.

We also prove it in the combinatorial way.

Proof

$$\begin{aligned}\sum_{i=0}^{n} i \cdot \binom{n}{i} &= \sum_{i=1}^{n} i \cdot \frac{n!}{i!(n-i)!} \\ &= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} \\ &= \sum_{i=1}^{n} n \cdot \binom{n-1}{i-1} \\ &= n \cdot \sum_{i=0}^{n-1} \binom{n-1}{i} \\ &= n \cdot 2^{n-1}\end{aligned}$$

■

3 Exercise 2.5.4

Prove by induction on n that

(a) $n^2 - 1$ is a multiple of 4 if n is odd,

(b) $n^3 - n$ is a multiple of 6 for every n.

Answer:

(a) **Proof** Let $n = 2k - 1$ for $k \in \mathbb{N}^*$.

The basic step, for $k = 1$, $(2 \cdot 1 - 1)^2 \equiv 1 \pmod 4$ Suppose it's true for $k - 1$,

$$\begin{aligned}(2k-1)^2 &\equiv (2(k-1)-1)^2 + 8(k-1) \pmod 4 && (3.1)\\ &\equiv 1 + 0 \pmod 4 && (3.2)\\ &\equiv 1 \pmod 4 && (3.3)\end{aligned}$$

Here $8(k-1) \equiv 0 \pmod 4$ in the Equation 3.2. Therefore, the proposition is held for all odd number n by the Induction Hypothesis. ■

(b) **Proof** The basic step, for $n = 1$, $1^3 - 1 \equiv 0 \pmod 6$ Suppose it's true for $n - 1$,

$$\begin{aligned}n^3 - n &\equiv (n-1)^3 - (n-1) + 3n(n-1) \pmod 6 && (3.4)\\ &\equiv 0 + 0 \pmod 6 && (3.5)\\ &\equiv 0 \pmod 6 && (3.6)\end{aligned}$$

Here $3n(n-1) \equiv 0 \pmod 6$ in the Equation 3.5 because there exists at least a 2 factor in $n(n-1)$. Hence, the proposition is held for all $n \in \mathbb{N}^*$ by the Induction Hypothesis. ■

4 Exercise 2.5.6

There is a class of all boys. We know that there are a boys who like to play chess, b who like to play soccer, c who like biking and d who like hiking. The number of those who like to play both chess and soccer is x.

There are y boys who like chess and biking, z boys who like chess and hiking, u who like soccer and biking, v boys who like soccer and hiking, and finally w boys who like biking and hiking. We don't know how many boys like, e.g.,chess, soccer and hiking, but we know that everybody likes at least one of these activities. We would like to know how many boys are in the class.

(a) Show by an example that this is not determined by what we know.

(b) Prove that we can at least conclude that the number of boys in the class is at most $a+b+c+d$, and at least $a+b+c+d-x-y-z-u-v-w$.

Answer:

Applying the Inclusion-Exclusion Principle, we have

$$\begin{aligned}|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D|-|A\cap B|-|A\cap C|-|A\cap D|-|B\cap C|-|B\cap D|-|C\cap D|\\ +|A\cap B\cap C|+|A\cap B\cap D|+|A\cap C\cap D|+|B\cap C\cap D|\\ -|A\cap B\cap C\cap D|\end{aligned}$$

where $|A\cup B\cup C\cup D| = n, |A| = a, |B| = b, |C| = c, |D| = d, |A\cap B| = x, |A\cap C| = y, |A\cap D| = z, |B\cap C| = u, |B\cap D| = v, |C\cap D| = w, |A\cap B\cap C| = o, |A\cap B\cap D| = p, |A\cap C\cap D| = q, |B\cap C\cap D| = r, |A\cap B\cap C\cap D| = s$.

(a) Suppose that $a = b = c = d = 2, x = y = z = u = v = w = 1$. If nobody likes more than two sports, namely $o = p = q = r = s = 0$, there are $n = a+b+c+d-x-y-z-u-v-w+o+p+q+r-s = 2$ boys in the class. If there is exactly a boy likes all four sports and others like less than three sports, namely $o = p = q = r = s = 1$, there are $n = a+b+c+d-x-y-z-u-v-w+o+p+q+r-s = 5$ boys in the class. Hence, the number of boys can not be determined.

(b) Because who likes all sports must like any there of them, $s \le o+p+q+r$.

$$\begin{aligned}n &= a+b+c+d-x-y-z-u-v-w+o+p+q+r-s\\ &\le a+b+c+d-x-y-z-u-v-w+s-s\\ &= a+b+c+d-x-y-z-u-v-w\end{aligned}$$

Thus, there are at least $a+b+c+d-x-y-z-u-v-w$ boys in the class.

Because everyone likes at least one sports, there are at most $a+b+c+d$ boys in the class.

5 Exercise 2.5.7

We select 38 even positive integers, all less than 1000. Prove that there will be two of them whose difference is at most 26.

Answer:

Proof We select 38 even positive integers as $1 < x_1 < x_2 < \cdots < x_{38} < 1000$. Prove it by contradiction. We suppose that $x_i - x_{i-1} \ge 27$ for all $i = 2, 3, \cdots, 38$. Namely,

$$\begin{aligned}x_2 - x_1 &\ge 27\\ x_3 - x_2 &\ge 27\\ &\cdots\\ x_{38} - x_{37} &\ge 27\end{aligned}$$

Summating them all, we have $999 = 1000 - 1 > x_{38} - x_1 \geq 27 \cdot 37 = 999$. There exists a contradiction with the condition. Thus, the proposition is held. ■

6 Exercise 2.5.8

A drawer contains 6 pairs of black, 5 pairs of white, 5 pairs of red, and 4 pairs of green socks.

(a) How many single socks do we have to take out to make sure that we take out two socks with the same color?

(b) How many single socks do we have to take out to make sure that we take out two socks with different colors?

Answer:

(a) 5 socks should be taken out to make sure there exist two socks with the same color because of Pigeonhole principle with four kinds of colors.

(b) 13 socks should be taken out to make sure there exists two socks with the different colors. In the worst case, the first twelve socks are the black socks. Because there is no more black sock, the thirteenth sock must be different with the first twelve ones.

7 Special Problem 1

Give a detailed derivation of $|T_j| = (n-1)!\frac{k}{j-1}$, using just the Addition and Multiplication Principles.

Answer:

The sample space S consists of all the permutations of $\{1, 2, \cdots, n\}$. An element $p = (i_1, i_2, \cdots, i_k, \cdots, i_n)$ is in the event T iff the following are satisfied.

(a) $1 \notin \{i_1, i_2, \cdots, i_k\}$

(b) if $i_j = 1 (k < j \leq n)$, $\min\{i_1, i_2, \cdots, i_{j-1}\}$ occurs in $\{i_1, i_2, \cdots, i_k\}$.

T_j denotes the set of the elements in T when 1 occurs in the position $j (k < j \leq n)$.

The calculation of $|T_j|$ is given as following.

Step 1 We choose $n - j$ numbers from $\{2, 3, \cdots, n\}$ to fill the position $j+1, j+2, \cdots, n$. There are P_{n-1}^{n-j} ways to do so.

Step 2 We find a position from $1, 2, \cdots, k$ to fill in the minimum of the remaining numbers after Step 1. There are k ways to do so.

Step 3 We choose $k-1$ numbers from the remaining numbers after Step 2 to fill the position $1, 2 \cdots k$. There are P_{j-2}^{k-1} ways to do so.

Step 4 We permute the remaining numbers to fill the position $k+1, \ldots, j-1$. There are P_{j-k-1}^{j-k-1} ways to do so.

The steps above are all independent. Applying the Multiplication Principle, we have

$$\begin{aligned}|T_j| &= P_{n-1}^{n-j} \cdot k \cdot P_{j-2}^{k-1} \cdot P_{j-k-1}^{j-k-1} \\ &= \frac{(n-1)!}{(j-1)!} \cdot k \cdot \frac{(j-2)!}{(j-k-1)!} \cdot (j-k-1)! \\ &= \frac{k}{j-1}(n-1)!\end{aligned}$$

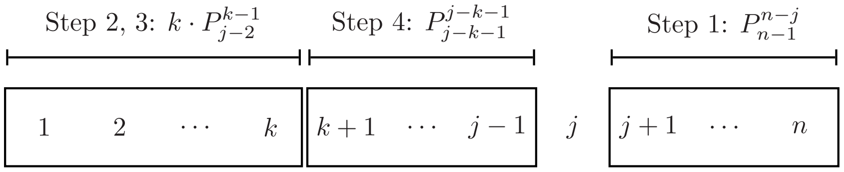

Figure 1: The sequence

8 Special Problem 2

In class, we derived a formula for $p_{n,k}$, the probability of success for finding the top person under strategy k for a random input permutation of $\{1, 2, \cdots, n\}$.

Now suppose we are less picky, and will be satisfied if the person selected is among the top two candidates. Let $q_{n,k}$ be the probability of achieving this goal under strategy k. Note that $q_{n,k} \geq p_{n,k}$.

(a) What is the value of $q_{4,2}$? You should write down all 24 permutations, and indicate which ones result in a success by strategy $k = 2$.

(b) Derive a mathematical expression for $q_{n,k}$. Check that it leads to the correct value for $n = 4, k = 2$, as derived in part (a).

Answer:

(a) Filter the solution by the simultation with Mathematica.

```
In[1] := qList[n_, k_] := Cases[Permutations[Range[n]], x_ /;
                          (m = Min @@ Take[x, k];
                          !FreeQ[Select[Drop[x, k], # < m &, 1], {1} | {2}])];
In[2] := qList[4, 2]
Out[2] = {{2, 3, 1, 4}, {2, 3, 4, 1}, {2, 4, 1, 3}, {2, 4, 3, 1},
>    {3, 2, 1, 4}, {3, 2, 4, 1}, {3, 4, 1, 2}, {3, 4, 2, 1}, {4, 2, 1, 3},
>    {4, 2, 3, 1}, {4, 3, 1, 2}, {4, 3, 2, 1}}
```

The list of all 24 permutations is below, where the ticked ones are the valid ones.

$1, 2, 3, 4$
$1, 2, 4, 3$
$1, 3, 2, 4$
$1, 3, 4, 2$
$1, 4, 2, 3$
$1, 4, 3, 2$

$2, 1, 3, 4$
$2, 1, 4, 3$
$2, 3, 1, 4 \quad \surd$
$2, 3, 4, 1 \quad \surd$
$2, 4, 3, 1 \quad \surd$
$2, 4, 1, 3 \quad \surd$

$3, 1, 2, 4$
$3, 1, 4, 2$
$3, 2, 1, 4 \quad \surd$
$3, 2, 4, 1 \quad \surd$
$3, 4, 1, 2 \quad \surd$
$3, 4, 2, 1 \quad \surd$

$4, 1, 2, 3$
$4, 1, 3, 2$
$4, 2, 1, 3 \quad \surd$
$4, 2, 3, 1 \quad \surd$
$4, 3, 1, 2 \quad \surd$
$4, 3, 2, 1 \quad \surd$

Count the valid permutations with Mathematica.

```
In[3] := q[n_, k_] := Count[Permutations[Range[n]], x_ /;
                         (m = Min @@ Take[x, k];
                         !FreeQ[Select[Drop[x, k], # < m &, 1], {1} | {2}])] / n!;
In[4] := q[4, 2]
Out[4] = 1 / 2
```

Thus, $q_{4,2} = 1/2$.

(b) If 1 is the one we chosen at the position j under strategy k, the number of the cases is $|T_j|$. If 2 is the one we chosen at the position j under strategy k, then 1 must occurs after 2, and there are $n - j$ positions after j. Then we permute the remaining $n - 2$ numbers as them in T_j. Thus,

$$|T'_j| = \frac{k}{j-1} \cdot (n-2)! \cdot (n-j)$$

Hence, by Addition Principle,

$$q_{n,k} = \frac{1}{n!} \sum_{j=k+1}^{n} \left(|T_j| + |T'_j|\right) = \frac{1}{n} \sum_{j=k+1}^{n} \frac{k}{j-1}(1 + \frac{n-j}{n-1})$$

Calculate $q_{4,2}$ with Mathematica.

```
In[5] := q2[n_, k_] := Sum[k / (j - 1) * (n - 1)!
                       + k / (j - 1) * (n - 2)! (n - j), {j, k + 1, n}] / n!;
In[6] := q2[4, 2]
Out[6] = 1 / 2
```

Validate the formula with Mathematica for $n = 1, 2, \cdots, 8, k = 1, 2, \ldots, n$.

```
In[7] := Table[q[n, k] == q2[n, k], {n, 1, 8}, {k, 1, n}]
Out[7]= {{True}, {True, True}, {True, True, True}, {True, True, True, True},
>    {True, True, True, True, True}, {True, True, True, True, True, True},
>    {True, True, True, True, True, True, True},
>    {True, True, True, True, True, True, True, True}}
```

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