Mathematics for Computer Science: Homework 3

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1 Exercise 3.8.5

Find the value of k for which $k\binom{99}{k}$ is largest.

Answer:

We want to compare two consecutive entries by getting the ratio of them.

$$\frac{(k+1)\binom{99}{k+1}}{k\binom{99}{k}} = \frac{99-k}{k}$$

So if $\frac{99-k}{k} < 1$, namely, $k < \frac{99}{2}$, $(k+1)\binom{99}{k+1}$ is greater than $k\binom{99}{k}$, and if $\frac{99-k}{k} > 1$, namely, $k > \frac{99}{2}$, $(k+1)\binom{99}{k+1}$ is smaller than $k\binom{99}{k}$. So when $k = \lceil \frac{99}{2} \rceil = 50$, $k\binom{99}{k}$ is largest.

2 Exercise 3.8.6

In city with a regular "chessboard" street plan, the North-South streets are called 1st Street, 2nd Street, $\cdots$, 20th Street, and the East-West streets are called 1st Avenue, 2nd Avenue, $\cdots$, 10th Avenue. What is the minimum number of blocks you have to walk to get from the corner of 1st Street and 1st Avenue to the corner of 20th Street and 10th Avenue? In how many ways can you get there walking this minimum number of blocks?

Answer:

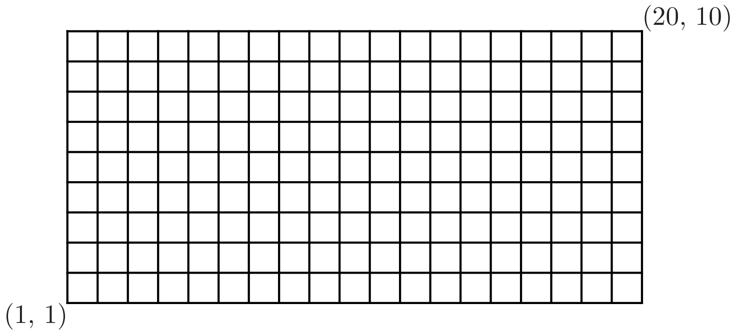

Figure 1: A regular "chessboard" street plan

We must go north 9 times and east 19 times in the shortest way. There are at least $9 + 19 = 28$ blocks we have to walk. At each street corner, we should decide to go north or east. It is equivalent to select 9 of $9 + 19$ corners at which we should go north. So there are $\binom{19+9}{9}$ different ways to walk in the shortest way.

3 Exercise 3.8.7

In how many ways can you read off the word MATHEMATICS from the following tables:

Answer:

'MATHEMATICS' is distributed through the diagonals. So we should walk from the left top corner to the right bottom corner in down or right direction each time. There is $\binom{5+5}{5} = 252$ ways to get 'MATHEMATICS' in the left-hand table, because we should walk down 5 times and right 5 times.

M	A	T	H	E	M
A	T	H	E	M	A
T	H	E	M	A	T
H	E	M	A	T	I
E	M	A	T	I	C
M	A	T	I	C	S

M	A	T	H		
A	T	H	E		
T	H		M	A	
H	E		A	T	I
	M	A	T	I	C
			I	C	S

Figure 2: MATHEMATICS tables

Let $f_{i,j}$ denote the number of ways to walk to (i,j) from $(1,1)$ which is the left top corner. We have the dynamic programming transfering equations as follows.

$$f_{i,j} = \begin{cases} f_{i-1,j} + f_{i,j-1} & \text{if the cell } (i,j) \text{ is in the table and nonempty} \\ 0 & \text{the cell } (i,j) \text{ is out of the table or empty in the table} \end{cases}$$

We calculate $f_{i,j}$ as follows.

M	A	T	H		
A	T	H	E		
T	H		M	A	
H	E		A	T	I
	M	A	T	I	C
			I	C	S

1	1	1	1	0	0
1	2	3	4	0	0
1	3	0	4	4	0
1	4	0	4	8	8
0	4	4	8	16	24
0	0	0	8	24	48

So the answer for the right-hand table is $f_{6,6} = 48$.

4 Exercise 3.8.8

Prove the following identities:

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}$$

Answer:

Proof

$$\begin{aligned} \sum_{k=0}^{m} (-1)^k \binom{n}{k} &= \binom{n}{0} + \sum_{k=1}^{m} (-1)^k \left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] \\ &= \binom{n-1}{0} - \left[\binom{n-1}{0} + \binom{n-1}{1}\right] + \left[\binom{n-1}{1} + \binom{n-1}{2}\right] \\ &\quad + \cdots + (-1)^m \left[\binom{n-1}{m-1} + \binom{n-1}{m}\right] \\ &= (-1)^m \binom{n-1}{m} \end{aligned}$$

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Proof

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m} &= \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{k!}{m!(m-k)!} \\
&= \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{k!}{m!(m-k)!}\frac{(n-m)!}{(n-m)!} \\
&= \sum_{k=0}^{n}\frac{n!}{m!(n-m)!}\frac{(n-m)!}{(n-k)!(m-k)!} \\
&= \binom{n}{m}\sum_{k=0}^{n}\binom{n-m}{n-k} \\
&= \binom{n}{m}\sum_{k=0}^{n-m}\binom{n-m}{k} \\
&= \binom{n}{m}2^{n-m}
\end{aligned}
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5 Exercise 3.8.10

In how many ways can you distribute n pennies to k children if each child is supposed to get at least 5?

Answer:

Suppose that the i-th child gets n_i pennies. We can formalize the problem as the number of the solutions of the equation

$$\sum_{i=1}^{k} n_i = n$$

where $n_i \geq 5$ for $i = 1, 2, \cdots, k$.

Let $n'_i = n_i - 5$ for $i = 1, 2, \cdots, k$. we rearrange the equation $\sum_{i=1}^{k} n_i = n$ to

$$\sum_{i=1}^{k} n'_i = n - 5k$$

where $n'_i \leq 0$ for $i = 1, 2, \cdots, k$. By Theorem 3.4.2, the number of ways to distribute $n - 5k$ pennies to k children is $\binom{n-5k+k-1}{k-1}$. So the answer is

$$\binom{n-4k-1}{k-1}$$

6 Exercise 3.8.14

Let n be a positive integer divisible by 3. Use Stirling's formula to find the approximate value of $\binom{n}{n/3}$.

Answer:

Because n is divisible by 3, let $n = 3k$.

$$\begin{aligned}\binom{n}{n/3} &= \frac{n!}{(n/3)!(2n/3)!} \\ &\sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{\frac{2}{3}\pi n}\left(\frac{n}{3e}\right)^{n/3}\sqrt{\frac{4}{3}\pi n}\left(\frac{2n}{3e}\right)^{2n/3}} \\ &= \frac{3}{2}\frac{1}{\sqrt{\pi n}}\frac{3^n}{2^{2n/3}} \\ &= \frac{\sqrt{3}}{2\sqrt{\pi k}}\frac{3^{3k}}{2^{2k}}\end{aligned}$$

where

$$\begin{aligned}n! &\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \\ (n/3)! &\sim \sqrt{\frac{2}{3}\pi n}\left(\frac{n}{3e}\right)^{n/3} \\ (2n/3)! &\sim \sqrt{\frac{4}{3}\pi n}\left(\frac{2n}{3e}\right)^{2n/3}\end{aligned}$$

by the Stirling's formula.

7 Special Problem 1

Let $a_1, a_2, \cdots, a_n$ be a sequence of real numbers. We say that the sequence is *unimodal* if there exists an $m \in \{1, 2, \cdots, n\}$ such that either (1) $a_1 \leq a_2 \leq \cdots \leq a_m$ and $a_m \geq a_{m+1} \geq \cdots \geq a_n$, or (2) $a_1 \geq a_2 \geq \cdots \geq a_m$ and $a_m \leq a_{m+1} \leq \cdots \leq a_n$. In LPV (pages 56-57), it is shown that for fixed n, $\binom{n}{k}(k = 1, 2, \cdots, n)$ is a unimodal sequence.

(a) Let $n > 1$ be any integer, and $0 < r < 1$. Prove that the sequence $b_1, b_2, \cdots, b_n$ is unimodal where $b_k = r^k(1-r)^{n-k}\binom{n}{k}$.

(b) Show that, for fixed n, the sequence $p_{n,k}(k = 1, 2, \cdots, n)$ in unimodal ($p_{n,k}$ is defined in Homework Set 2 Special Problem 2).

Answer:

(a) We want to compare two consecutive entries by getting the ratio of them.

$$\begin{aligned}\frac{b_{k+1}}{b_k} =& \frac{r^{k+1}(1-r)^{n-k-1}\binom{n}{k+1}}{r^k(1-r)^{n-k}\binom{n}{k}} \\ =& \frac{r(n-k)}{(1-r)(k+1)}\end{aligned}$$

If $k \leq \lfloor nr + r - 1 \rfloor$, $b_{k+1} \geq b_k$. And $k \geq \lceil nr + r - 1 \rceil$, $b_{k+1} \leq b_k$. There exists the largest b_k whose k is nearby $nr + r - 1$. All items before the largest b_k is increasing monotonously. All items after the largest b_k is decreasing monotonously. So b_k is unimodal.

(b)

$$p_{n,k} = \frac{1}{n!}\sum_{j=k+1}^{n} |T_j| = \frac{k}{n}\sum_{j=k}^{n-1}\frac{1}{j}$$

$$\begin{aligned} p_{n,k+1} - p_{n,k} &= \frac{k+1}{n}\sum_{j=k+1}^{n-1}\frac{1}{j} - \frac{k}{n}\sum_{j=k}^{n-1}\frac{1}{j} \\ &= \frac{1}{n}\left(\sum_{j=k+1}^{n-1}\frac{1}{j} - 1\right) \end{aligned}$$

The difference of consecutive entries $p_{n,k+1} - p_{n,k}$ is decreasing monotonously. So The $p_{n,k}$ is convex, namely $p_{n,k}$ is unimodal.

8 Special Problem 2

A binary string α is said to be *balanced* if the number of 1's in the first half of α is exactly equal to the number of 1's in the second half. (For example, 00100110 is *not* balanced since there are two 1's in the second half and only one 1 in the first half; on the other hand, the string 01001000 is balanced.) Prove that there are exactly $\binom{2n}{n}$ balanced strings of length $2n$.

Answer:

We enumerate k which is the number of 1's in the first half of the string, and k is also the number of 1's in the second half. There is $\binom{n}{k}$ ways to select k bits of n bits of the binary string to set to 1. So $\binom{n}{k}^2$ ways to select exactly k bits in the first half and k bits in the second half to set to 1. The answer is

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

by the formula (3.4) in LPV.

9 Special Problem 3

Consider a sequence of $2n$ people in a line at a cashier. Suppose n of the people pay 1 yuan each and n of the people get 1 yuan each. A *paying pattern* is a binary sequence $\sigma = a_1a_2\cdots a_{2n}$ with exactly n 1's and n 0's; the interpretation is that $a_j = 1$ if person j pays 1 yuan, and $a_j = 0$ otherwise. Note that there are exactly $\binom{2n}{n}$ paying patterns. Prove that the number of paying patterns in which the cashier never goes in debt (i.e., at every stage at least as many people have paid in 1 yuan as were paid out 1 yuan) is equal to $\binom{2n}{n} - \binom{2n}{n+1}$. (**Hint**: Show a one-to-one correspondence between paying patterns where at some stage the cashier goes at least 1 yuan in debt and all binary sequences of length $2n$ with exactly $n+1$ 1's.)

Answer:

Consider the sequences $\tau = b_1b_2\cdots b_{2n}$ which consists of n '1's and n '-1's. We call it valid iff $\sum_{i=1}^{k} b_i \geq 0$ for all $k = 1, 2, \cdots, 2n$. Obviously, a valid τ is equivalent to a valid σ mentioned in the problem description.

For each invalid sequence τ, there exists k which is the first position satisfying $\sum_{i=1}^{k} b_i = -1 < 0$. k must be an odd number, and $\sum_{i=1}^{k-1} b_i = 0$. We reverse the first k elements of τ, namely convert -1 to 1 and 1 to -1. Now we get the sequence τ' which consists of $n+1$ '1's and $n-1$ '-1's.

Given a sequence $\tau' = b'_1 b'_2 \cdots b'_{2n}$ which consists of $n+1$ '1's and $n-1$ '-1's. Because the number of '1's is more than '-1's, there exists the first k such than $\sum_{i=1}^{k} b'_i > 0$. Reversing the first k elements in τ', we get an invalid sequence τ, which consists of n '1's and n '-1's.

Till now, we construct a one-to-one correspondence between the invalid sequences and the sequences which consist of $n+1$ '1's and $n-1$ '-1's So the number of the invalid sequences is $\binom{2n}{n+1}$. There is $\binom{2n}{n}$ τ sequences in total. Thus, the number of the valid sequences is $\binom{2n}{n} - \binom{2n}{n+1}$.