Mathematics for Computer Science: Homework 1

Instructed by *Andrew C. Yao*

Due on Mar 5, 2008

Botao Hu J72 2007011292

hupo001@gmail.com

Contents

1 Exercise 1.3.1

Under the correspondence between numbers and subsets described above, which numbers correspond to (a) subsets with 1 element, (b) the whole set? (c) Which sets correspond to even numbers?

Answer:

(a) The numbers which correspond to the subsets with 1 element must be a power of two because there is exactly one bit "1" in its binary representation.

(b) The binary representation of the whole set is composed of all "1" bits. Therefore, the answer is $2^n - 1$ where n is the element number of the whole set.

(c) If the number is even, the last bit in its binary representation is 0. Thus, the last element must not be contained in the corresponding subset.

2 Exercise 1.3.3

Show that a nonempty set has the same numbers of odd subsets (i.e., subsets with an odd number of elements) as even subsets.

Answer:

Proof Denote n as the number of elements in the nonempty set. Let odd_n be the odd subsets of the nonempty set with size n. Define $even_n$ similarly.

Obviously, $odd_1 = \{1\}, \quad even_1 = \emptyset, \quad |odd_1| = |even_1|$.

odd_n must contain odd_{n-1}. In the other hand, odd_n can be obtained by adding n-th element into each sets in $even_{n-1}$. We have

$$odd_n = odd_{n-1} \cup even_{n-1}, \quad |odd_n| = |odd_{n-1}| + |even_{n-1}|$$

Similarly, we have

$$even_n = odd_{n-1} \cup even_{n-1}, \quad |even_n| = |odd_{n-1}| + |even_{n-1}|$$

Hence, we get $|odd_n| = |even_n|$ for each $n \in \mathbb{N}^*$ by the Induction Principle. ■

3 Exercise 1.5.1

Draw a tree illustrating the way we counted the number of strings of length 2 formed from the characters a, b, and c, and explain how it gives the answer. Do the same for the more general problem when $n = 3, k_1 = 2, k_2 = 3, k_3 = 2$.

Answer:

The number of strings is the number of the leaves of the tree. The strings can be formed by concatenating the letters in the path from root to the leaf. Thus, there is a bijection between the paths to the leaves and the strings.

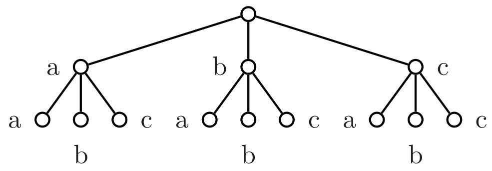

Figure 1: There are 3×3 leaves in the tree.

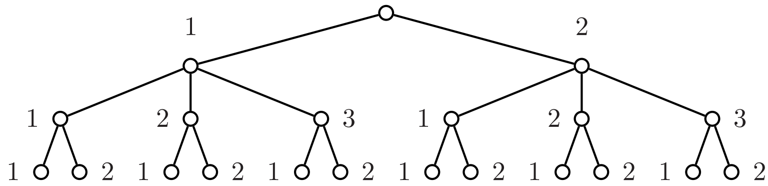

Figure 2: There are $k_1 k_2 k_3 = 2 \times 3 \times 2$ leaves in the tree.

4 Exercise 1.5.6

We have 20 kinds of presents; this time, we have a large supply of each kind. We want to give presents to 12 children. Again, it is not required that every child gets something; but no child can get two copies of the same present. In how many ways can we give presents?

Answer:
For each child, there are 2^{20} ways being given presents. There are $(2^{20})^{12}$ ways to distribute the presents.

5 Exercise 1.8.10

List all subsets of $\{a, b, c, d, e\}$ containing $\{a, e\}$ but not containinig c.

Answer:
The appearance of a, e, c is determinate. Just enumerate the subsets of $\{b, d\}$. The answer is

$$\{a, e\}, \{a, b, e\}, \{a, d, e\}, \{a, b, d, e\}$$

6 Exercise 1.8.19

Prove that for any three sets A, B, C, $((A \setminus B) \cup (B \setminus A)) \cap C = ((A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$.

Answer:

Proof

$$((A \setminus B) \cup (B \setminus A)) \cap C \tag{6.1}$$
$$= ((A \cup B) \setminus (A \cap B)) \cap C \tag{6.2}$$
$$= ((A \cup B) \cap C) \setminus ((A \cap B) \cap C) \tag{6.3}$$
$$= ((A \cap C) \cup (A \cup B)) \setminus (A \cap B \cap C) \tag{6.4}$$

(6.2) can be obtained by the definition of the symmetric difference. (6.3) is based on the distributivity of intersection of sets over difference of sets. (6.4) is based on the distributivity of intersection of sets over union of sets. ■

7 Exercise 1.8.24

How many bits does 10^{100} have if written in base 2?

Answer:

We want to find k such that $2^{k-1} \leq 10^{100} < 2^k$. So

$$k = \lfloor \log_2 10^{100} \rfloor + 1 = 333$$

8 Exercise 1.8.26

Find the number of all 20-digit integers in which no two consecutive digits are the same.

Answer:

The first digit has 9 choices from 1 to 9 because there is no leading zero in the integer. The i-th digit has 9 choices of the digit which is different from the $(i-1)$-th digit for all $i > 1$. Namely, we have $a_1 = 9, \quad a_{i+1} = 9 \cdot a_i$. The answer is $a_{20} = 9^{20}$.

9 Exercise 1.8.29

What is the number of ways to color n objects with 3 colors if every color must be used at least once?

Answer:

The case that objects are labeled. The negative problem is to color n objects with exactly 1 or 2 colors. Because of Inclusion-Exclusion Principle, the answer is obvious:

$$3^n - 2^n \binom{3}{2} + 1^n \binom{3}{1}$$

The case that objects are unlabeled. It's equivalent to find the number of the solutions for the integer equations below.

$$x + y + z = n, \quad x, y, z \geq 1$$

It's easy to solve under the condition $x, y, z \geq 0$. The number of the solutions is $\binom{n+3-1}{2}$.
Under $x, y, z \geq 1$, let $x' = x - 1, y' = y - 1, z' = z - 1$. Thus, the problem is reduced to

$$x' + y' + z' = n + 1 + 1 + 1, \quad x', y', z' \geq 0$$

The answer is

$$\binom{n-3+3-1}{2} = \binom{n-1}{2}$$

10 Exercise 1.8.30

Draw a tree for Alice's solution of enumerating the number of ways 6 people can play chess, and explain Alice's argument using the tree.

Answer:

The method is simple that each time, let the person with the smallest index in the unsettled people choose his partner. For example, the plan $1, 3, 2, 5, 4, 6$ can be obtained by follows.

Let the 1-numbered person who is the smallest in $\{1, 2, 3, 4, 5, 6\}$ choose a partner in $\{2, 3, 4, 5, 6\}$. In this example, the 1-numbered chooses the 3-numbered. $\{1, 3\}$ is shown in the tree edge from $\{1, 2, 3, 4, 5, 6\}$ to $\{2, 4, 5, 6\}$. Then let the 2-numbered person who is the smallest in the remaining set $\{2, 4, 5, 6\}$ choose a partner in $\{4, 5, 6\}$. In this example, the 2-numbered chooses the 5-numbered. $\{2, 5\}$ is shown in the tree edge from $\{2, 4, 5, 6\}$ to $\{4, 6\}$. At last, the two remaining people make a pair $\{4, 6\}$. So there are $5 \times 3 \times 1 = 15$ ways to settle players.

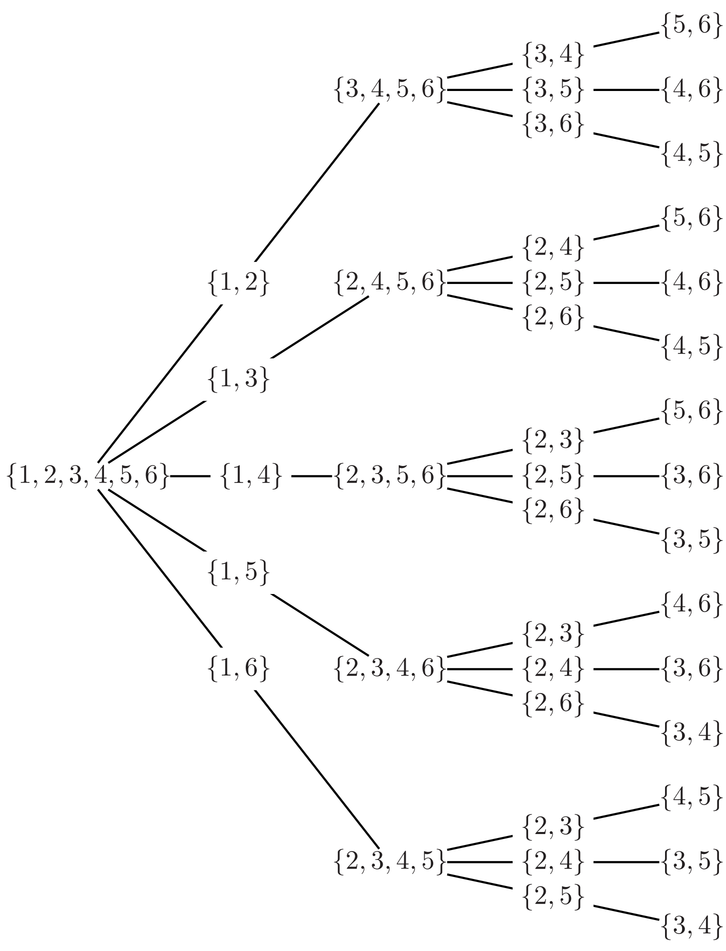

Figure 3: There are $5 \times 3 \times 1 = 15$ ways to settle players.

11 Exercise 2.1.8

Prove that the sum of the first n squares $\sum_{i=1}^{n} i^2$ is

$$\frac{n(n+1)(2n+1)}{6}$$

Answer:

Proof For $n = 1$, it's true. For $n > 1$,

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n-1} i^2 + n^2 \tag{11.1}$$

$$= \frac{(n-1)n(2n-1)}{6} + n^2 \tag{11.2}$$

$$= \frac{2n^3 + 3n^2 + n}{6} \tag{11.3}$$

$$= \frac{n(n+1)(2n+1)}{6} \tag{11.4}$$

Here we suppose it's true for $n-1$ so that $\sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}$. So (11.2) can be obtained from (11.1).

The proposition is obvious based on the Induction Hypothesis. ■

12 Exercise 2.1.9

Prove that the sum of the first n powers of 2 (starting with $1 = 2^0$) is $2^n - 1$.

Answer:

It's true obviously for $n = 1$. For $n > 1$,

$$\sum_{i=0}^{n-1} 2^i = \sum_{i=0}^{n-2} 2^i + 2^{n-1} \tag{12.1}$$

$$= 2^{n-1} - 1 + 2^{n-1} \tag{12.2}$$

$$= 2^n - 1 \tag{12.3}$$

The proposition is hold for all $n \in \mathbb{N}^*$ by the Induction Hypothesis.

13 Exercise 2.1.11

Use induction on n to prove the "handshake theorem" (the number of handshakes between n people is $\frac{n(n-1)}{2}$).

Answer:

Denote p_n as the number of handshakes between n people. Obviously, $p_1 = 0$. Consider the case that $n > 1$. Suppose till now there are $n-1$ people finished their handshakes and $p_{n-1} = \frac{(n-1)(n-2)}{2}$. After joining a new person who will shake hands with all $n-1$ people who came before him, hence the total number of handshakes will be increased by $n-1$.

$$p_n = p_{n-1} + n - 1 = \frac{(n-1)(n-2)}{2} + n - 1 = \frac{(n-1)n}{2}$$

14 Exercise 2.1.12

Read carefully the following induction proof:

> ASSERTION: *$n(n+1)$ is an odd number for every n.*
> PROOF: Suppose that this is true for $n-1$ in place of n; we prove it for n, using the induction hypothesis. We have
>
> $$n(n+1) = (n-1)n + 2n.$$
>
> Now here $(n-1)n$ is odd by the induction hypothesis, and $2n$ is even. Hence $n(n=1)$ is the sum of an odd number and an even number, which is odd.

The assertion that we proved is obviously wrong for $n = 10$: $10 \cdot 11 = 110$ is even. What is wrong with the proof?

Answer:

The base case for induction is wrong. When $n = 1$, $1 \times 2 = 2$ is an even number.

15 Exercise 2.1.13

Read carefully the following induction proof:

> ASSERTION: *If we have n lines in the plane, no two of which are parallel, then they all go through one point.*
> PROOF: The assertion is true for one line (and also for 2, since we have assumed that no two lines are parallel). Suppose that it is true for any set of $n-1$ lines. We are going to prove that it is also true for n lines, using this induction hypothesis.
>
> So consider a set $S = \{a, b, c, d, \ldots\}$ of n lines in the plane, no two of which are parallel. Delete the line c: then we are left with a set S' of $n-1$ lines, and obviously, no two of these are parallel. So we can apply the induction hypothesis and conclude that there is a point P such that all the lines in S' go through P. In particular, a and b go through P, and so P must be the point of intersection of a and b.
>
> Now put c back and deleting d, to get a set S'' of $n-1$ lines. Just as above, we can use the induction hypothesis to conclude that these lines go through the same point P': but just as above, P' must be the point of intersection of a and b. Thus $P' = P$. But then we see that c goes through P. The other lines also go through P (by the choice of P), and so all the n lines go through P.

But the assertion that we proved is clearly wrong: where is the error?

Answer:

In the induction step, if $|S| = 3$, there doesn't exist d. Thus the induction step haven't proven the case $n = 3$. In fact, three lines can have three intersection points.

16 Optional Problem 1

For a group of n people each with random birthday, let p_n be the probability that there exist three people with the same birthday.

(a) Derive a mathematical expression of p_n in terms of n.

(b) What is the numerical value of p_9?

(c) Can you estimate n when p_n first becomes greater than 1/2?

Answer:

Denote m as the number of the total days of one year and n as the number of people. We call the sequence formed by n people's birthdays as L. Consider the negative problem: the probability that each day occurs at most twice in L, denoted as q_n. So we have $p_n = 1 - q_n$.

(a) Enumerate i as the number of the days which occur exactly twice in L. Step 1, to choose $n-i$ days from m days as all the distinct days in L. Step 2, to choose i days from $n-i$ distinct days as the repeated days in L. Till now, the number of the combination for L is gotten. Step 3, to multiply it by the number of the permutations of the current L in which i days repeat twice. The number is $\frac{n!}{(2!)^i(1!)^{n-i}}$. Therefore, we have

$$q_n = \frac{1}{m^n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{m}{n-i} \binom{n-i}{i} \frac{n!}{(2!)^i} \right]$$

(b) Calculate the numerical value of p_9 with Mathematica as follows.

```
In[1] := p[m_, n_] :=
          1 - Sum[Binomial[m, n - i] * Binomial[n - i, i] * n! / 2^i,
                {i, 0, Floor[n / 2]}] / m^n;
In[2] := p[365, 9]
Out[2] = 7847041916637793/12600938935845015625
In[3] := N[%]
Out[3] = 0.000622735
```

So $p_9 \approx 0.000622735$

(c) Seek n that $p_n > 0.5$ by the implementation with Mathematica.

```
In[4] := NestWhile[# + 1 &, 1, p[365, #] <= 0.5 &]
Out[4] := 88
```

Hence when $n \geq 88$, $p_n > 0.5$.

The formula in this problem is easy to figure out in the combination way because the requirement is to find only three people with the same birthday. But how about r people with the same birthday? The following is try to generalize the problem.

Consider the generation function $G(x)$:

$$G(x) = \left(\sum_{i=0}^{r-1} \frac{x^i}{i!} \right)^m = \sum_{i=0}^{\infty} q_i \frac{m^i x^i}{i!} \tag{16.1}$$

Here the item $\sum_{i=0}^{r-1} \frac{x^i}{i!}$ denotes each day there are at most $r-1$ people has the same birthdays. The answer is $1-q_i$.

Validate it with Mathematica.

```
In[5] := p[n_, m_, r_] :=
         1 - n! / m^n * SeriesCoefficient[Sum[x^i / i!,
                                      {i, 0, r - 1}]^m, {x, 0, n}];
In[6] := p[9, 365, 3]
Out[6] = 7847041916637793/12600938935845015625
```

To give the recurrence formula, we derivate $G(x)$.

$$G'(x) = m \left(\sum_{i=0}^{r-1} \frac{x^i}{i!} \right)^{m-1} \left(\sum_{i=0}^{r-2} \frac{x^i}{i!} \right) = \sum_{i=1}^{\infty} q_i \frac{m^i x^{i-1}}{(i-1)!} \tag{16.2}$$

Compared with left sides of Equation 16.1 and Equation 16.2, we have

$$m \left(\sum_{i=0}^{r-2} \frac{x^i}{i!} \right) G(x) = \left(\sum_{i=0}^{r-1} \frac{x^i}{i!} \right) G'(x)$$

$$\Rightarrow \left(\sum_{i=0}^{r-2} \frac{x^i}{i!} \right) \left(\sum_{i=0}^{\infty} q_i \frac{m^i x^i}{i!} \right) = \left(\sum_{i=0}^{r-1} \frac{x^i}{i!} \right) \left(\sum_{i=0}^{\infty} q_{i+1} \frac{m^i x^i}{i!} \right)$$

Compared with the coefficients, we have

$$q_n = \sum_{i=0}^{r-2} \left[\frac{1}{m^i} \binom{n-1}{i} - \frac{1}{m^{i+1}} \binom{n-1}{i+1} \right] q_{n-i-1} \quad (n \geq r) \tag{16.3}$$

$$q_1 = q_2 = \ldots = q_{r-1} = 1 \tag{16.4}$$

Equation 16.3 and Equation 16.4 gives the recurrence formula of q_n. Now validate it with Mathematica.

```
In[7] := q[n_, m_, r_] := If[n < r, 1,
        Sum[q[n - i - 1, m, r](Binomial[n - 1, i] / m^i
            - Binomial[n - 1, i + 1] / m^(i + 1)), {i, 0, r - 2}]];
In[8] := 1 - q[9, 365, 3]
Out[8] = 7847041916637793/12600938935845015625
```

17 Optional Problem 2

A 4×4 box is filled with integers $1, 2, ..., 15$ with the lower-right corner initially left empty. At any time there is exactly one cell empty. At each step you may move to the empty cell one of its adjacent number. Question: Prove that the second configuration cannot be reached from the first configuration.

Answer:

Define the *permutation* as the sequence which is obtained by concatenating all the rows of the current configuration in the upper-lower order and removing the empty piece.

We predicate that the invariant under any valid move is the *parity* of the number of the pairs of the pieces in reversed order in the current permutation (called *the reversed-pair number*) *plus* the row number of the empty piece. We call the invariant *the state function*.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 4: Configuration 1, 2

Theorem 17.1 *Under any valid move, the parity of the state function of one configuration is always fixed.*

Proof The case for the left move or the right move.

The row number of the empty piece will not be changed. Because the move only swaps the empty piece and the row-adjacent piece, the permutation will not be changed. Therefore, the parity of the state function will not be changed.

The case for the up move or down move.

Here we only consider the case for the down move. The row number of the empty piece will be changed by one.

Let be p the permutation. Denote p_i as the swapped piece and $p_{i+1}, \ldots, p_{i+3}$ as all the pieces between the swapped piece and the empty piece in the permutation.

For some piece $p_{i+j} (1 \leq j \leq 3)$, if $p_i > p_j$, the reversed-pair number will decrease by one after a move. Otherwise, if $p_i < p_j$, the reversed-pair number will increase by one after a move. There are 3 pieces changing the reversed-pair number, so the reversed-pair number will be changed by an odd value.

Therefore, the parity of the state function will not be changed. ■

The first configuration's state function is $0+4$. The second configuration's state function is $1+4$. Due to our predication above, the difference of their parities causes the puzzle are impossible to resolve, no matter how many moves are made.

Reference: Wikipedia, http://en.wikipedia.org/wiki/Fifteen_puzzle

Acknowledgement: Thank Zhe Yang (2007011295) for LaTeX's guide and discussion about the generation function.