Mathematics for Computer Science: Homework 7

Instructed by *Andrew C. Yao*

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Contents

1 Exercise 7.3.5

Does there exist a graph with the following degrees: (a) 0, 2, 2, 2, 4, 4, 6; (b) 2, 2, 3, 3, 4, 4, 5.

Answer:

(a) There exists a vertex v such that $\deg(v) = |V(G)| - 1 = 6$. Namely, the vertex v connects to each other vertex. But there is an isolate vertex. There does not exist such graph.

(b) The total summation of the degrees of all vertices must be even number. But the sum $2+2+3+3+4+4+5 = 23$ is an odd number. There does not exist such graph.

2 Exercise 7.3.9

Prove that at least one of G and $\overline{G}$ is connected.

Answer:

Without loss of generality, suppose that G is unconnected. Suppose G has k maximal connected components denoted by $C_1, C_2, \cdots, C_k$ $(k > 1)$. Let $u \in C_i$, $v \in C_j$. If $i \neq j$, there exists an edge $(u, v) \in E(\overline{G})$. Otherwise, if $i = j$, there exists a vertex $w \in C_k (k \neq i)$ such that there exist the edge (u, w) and (v, w) in $E(\overline{G})$. Thus, for any vertices u, v, there exists a path connected them in $\overline{G}$.

3 Exercise 7.3.13

There are $(m-1)n+1$ people in a room. Show that either there are m people who mutually do not know each other, or there is a person who knows at least n others.

Answer:

Suppose that each person knows at most $n-1$ people and there are at most $m-1$ people who mutually do not know each other. The total degree of these $m-1$ people is at most $(n-1)(m-1)$. But there are $(n-1)(m-1)+1$ people remaining. So there should be one person who doesn't know anybody among these $m-1$ people. It's conflictive to the assumption: at most $m-1$ people. Thus we find the contradiction.

4 Exercise 2 from "The Story of Louis Pósa"

Prove that there is no Hamiltonian circuit in each of the following graphs:

Answer:

Lemma 4.1 (Hamiltonian necessary condition) *Denote the number of connected components of a graph G by $c(G)$.*

(a) If G has a hamiltonian circuit, for any non-empty subset $S \subset V(G)$, $c(G - S) \leq |S|$.

(b) If G has a hamiltonian path, for any non-empty subset $S \subset V(G)$, $c(G - S) - 1 \leq |S|$.

Proof (a) Because there exist $c(G-S)$ connected components and we need to connect them and S alternatively to form a cycle, there would be $c(G-S)$ edges entering the subset S. Thus, there are at least $c(G-S)$ vertices in S connected to these $c(G-S)$ connected components through those entering edges.

(b) Because there exist $c(G-S)$ connected components and we need to connect them and S alternatively to form a path, there would be $c(G-S)-1$ edges entering the subset S. Thus, there are at least $c(G-S)-1$ vertices in S connected to these $c(G-S)$ connected components through those entering edges. ■

Apply Lemma 4.1 as the following.

(a) $S=\{1,4\}, c(G-S)=|\{\{3\},\{5\},\{6\},\{2\}\}|=4>|S|=2$

(b) $S=\{2,4\}, c(G-S)=|\{\{1\},\{7\},\{3,5,6,8\}\}|=3>|S|=2$

(c) $S=\{3,6,9,12\}, c(G-S)=|\{\{4,5\},\{1,2,8\},\{10,11\},\{7\},\{13,14,15\}\}|=5>|S|=4$

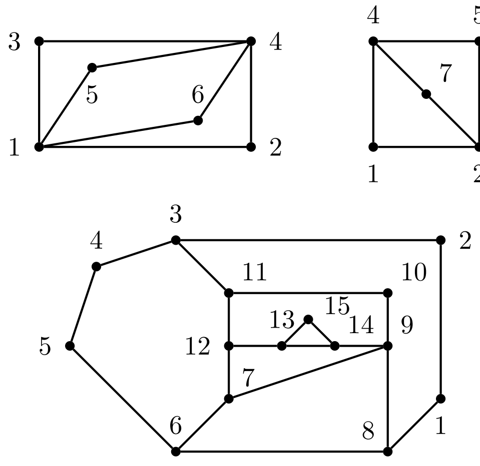

Figure 1: (a), (b) and (c)

The following Mathematica program can be used to find the subset S.

```
In[1] := <<Combinatorica‘;
In[2] := g = {{1,2},{1,4},{2,3},{2,5},{3,6},{4,5},{5,6},{4,7},{7,2},{5,8},{8,3}};
In[3] := Select[Complement[Subsets[Range[Max[g]]], {{}}], Length[#] <
             Length[ConnectedComponents[DeleteVertices[FromUnorderedPairs[g], #]]]&, 1]
Out[3] = {2, 4}
```

5 Exercise 4 from “The Story of Louis Pósa”

Which of the following graphs have Hamiltonian circuits, and which have only Hamiltonian paths?

Answer:

(a) Although the graph of (a) satisfies Lemma 4.1, using the brute force program below, we found that there doesn’t exist Hamiltonian cycle for (a). But (a) has Hamiltonian path $P=\{1,2,6,7,3,5,4\}$.

(b) The hamiltonian cycle for (b): $C=\{1,2,3,4,8,7,6,10,11,12,16,15,14,13,9,5,1\}$

(c) The hamiltonian cycle for (c): $C=\{1,2,3,4,5,10,17,9,16,8,20,7,19,14,15,11,12,13,18,6,1\}$

The following Mathematica program can be used to find the Hamiltonian cycle.

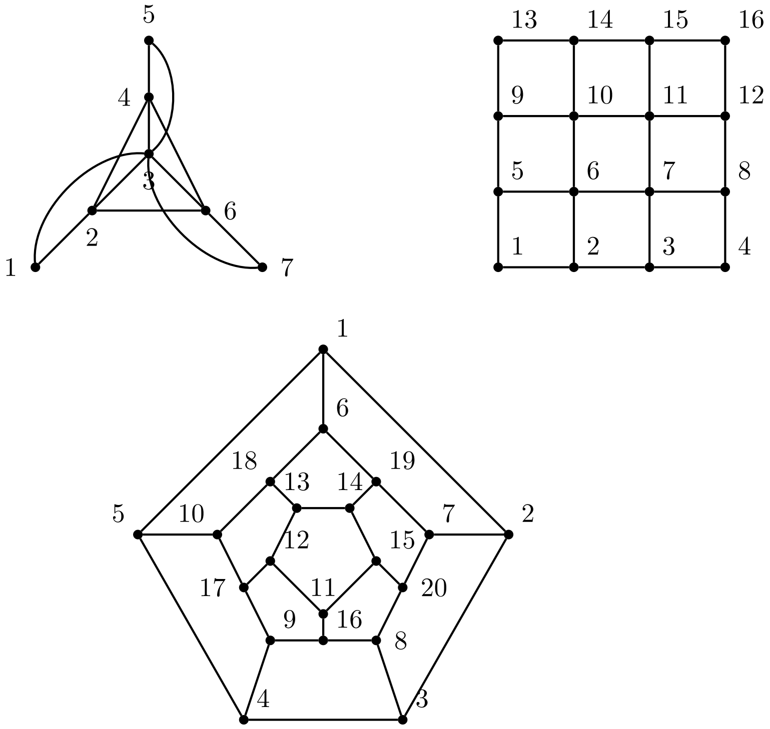

Figure 2: (a), (b) and (c)

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In[1] := <<Combinatorica`;
In[2] := g = {{1,2},{2,3},{3,4},{4,5},{7,6},{6,3},{2,4},{2,6},{4,6},{1,3},{5,3},{7,3}};
In[3] := HamiltonianCycle[FromUnorderedPairs[g]]
Out[3] = {}
```

6 Exercise 5 from "The Story of Louis Pósa"

Six points in general position are given in three-dimensional space (that is, no 3 are collinear and no 4 are coplanar). The $C(6,2) = 15$ segments joining them in pairs are colored individually either red or blue at random. Prove that some triangle has all its sides the same color.

Answer:

We build a graph model for this problem: map all points to the vertices, and if there is a side colored red, there is an edge between the corresponding vertices. What we need to prove is that there exists a K_3 contained in either G or $\overline{G}$. Because $\deg_G(u) + \deg_{\overline{G}}(u) = 5$ for a certain vertex u, we suppose $deg_G(u) \geq 3$ without loss of generity by Pigeonhole Principle. If there exist the vertices v and w adjacent to u such that $(v, w) \in E(G)$, we are done. Otherwise, there exist at least three vertices adjacent to u such that they are not adjacent mutually. Thus, they form K_3 in $\overline{G}$.

7 Exercise 6 from "The Story of Louis Pósa"

A set of moves in chess which takes a knight successively through all 64 squares is called a knight's tour. If the knight can go from the last square to the first one in one move, and thus go all around again, the tour is called re-entrant. A re-entrant knight's tour corresponds to a Hamiltonian circuit in a graph which has a vertex for each square and an edge joining the vertices representing squares X and Y if and only if a knight can go from X to Y in one move. Show that there is no re-entrant knight's tour on any chessboard which has dimensions 4 by n, n a natural number.

Answer:

Suppose that $4 \times n$ board has a closed knight's tour. If the cell (i, j) statisfies that $i + j$ is an odd integer, we color it black. Otherwise, if $i + j$ is an even integer, we color it white. Obviously, the knight must alternately jump between white and black cells.

If the cell (i, j) statisfies that $i = 1$ or $i = 4$, we mark it A. Otherwise, if it statisfies that $i = 2$ or $i = 3$, we mark it B. Obviously, the number of the cells marked A is the same as the cells marked B. From a cell marked A, the knight must next jump to a cell marked B. The cells marked A must be separated by the cells marked B. Because the knight must visit every square, and the number of the cells marked A is equal to the cells marked B, the knight must alternately jump between A-marked and B-marked cells.

We suppose the first cell which the knight goes to would be a cell of white and A without loss of generality (if it's not, we swap A and B or white and black). The second cell must be an element of the set of the cells colored black and marked B. The third cell must be an element of the set of the cells colored white and marked A, which is the same as the set in the first step. The fourth cell must be an element of the set of the cells colored black and marked B, which is the same as the set in the second step. And so on.

It imples that the set of the cells marked A has the same elements as the set of the cells colored white. and the set of the cells marked B has the same elements as the set of the cells colored black. Obviously, it's not true.

Reference: Allen J. Schwenk, *Which Rectangular Chessboards Have a Knight's Tour?*, Mathematics Magazine, Vol. 64, No. 5 (Dec., 1991), pp. 325-332

8 Special Problem 1

Do Exercise 10 in the article "The Story of Louis Pósa". Let $A_0, A_1, A_2, \cdots, A_{2n-1}$ denote, in cyclic order, the vertices of a regular $2n$-gon. Let all the sides and diagonals be drawn to give graph G. Prove that every Hamiltonian circuit of G must contain two edges which are parallel lines in the diagram.

Answer:

All the indices in the following description is under the operator modulo $2n$. Let $v_0, v_1, \cdots, v_{2n-1}$ be the index sequence of a Hamiltonian circuit of G. Define $v_{2n} = v_0$. Obviously, the edge (A_i, A_j), (A_{i-1}, A_{j+1}), $\cdots$, (A_{i-k}, A_{j+k}), $\cdots$ are mutually parallel. So if (v_i, v_{i+1}) is parallel to (v_j, v_{j+1}), then $v_i - v_j \equiv v_{j+1} - v_{i+1} \equiv k \pmod{2n}$, namely, $v_i + v_{i+1} \equiv v_j + v_{j+1} \pmod{2n}$.

Now we need to prove that there exists $i, j \in \{0, 1, \cdots, 2n-1\}$ such that $v_i + v_{i+1} \equiv v_j + v_{j+1} \pmod{2n}$.

$$\sum_{i=0}^{2n-1} v_i + v_{i+1} \equiv 2 \sum_{i=0}^{2n-1} i \equiv 2n(2n-1) \equiv 0 \pmod{2n}$$

To derive the contradiction, suppose all the values of $v_i + v_{i+1}$ are different. Namely, $\{v_i + v_{i+1} : i = 0, 1, \cdots, 2n-1\} = \{0, 1, \cdots, 2n-1\}$

$$\sum_{i=0}^{2n-1} v_i + v_{i+1} \equiv \sum_{i=0}^{2n-1} i \equiv n(2n-1) \equiv -n \pmod{2n}$$

Because $\sum_{i=0}^{2n-1} v_i + v_{i+1} \equiv 0 \not\equiv -n \equiv \sum_{i=0}^{2n-1} v_i + v_{i+1} \pmod{2n}$, the contradiction occurs.

9 Special Problem 2

Let $G_n = (V, E)$, where V is the set of all $n!$ permutations of $\{1, 2, \cdots, n\}$, and there is an edge $\{\sigma, \rho\} \in E$ if and only if σ can be obtained from ρ by the interchange of two numbers. For example, for $n = 5$, there is an edge between $(1, 3, 5, 4, 2)$ and $(1, 3, 2, 4, 5)$ since they differ only in the interchange of 2 and 5, but there is no edge between $(1, 3, 5, 4, 2)$ and $(1, 5, 2, 4, 3)$. Show that, for all $n > 2$, G_n has a Hamiltonian cycle.

Answer:

In fact, we can construct the Hamiltonian cycle with a more strict condition - only swap the adjacent elements in the permutation. We suppose that we have already obtained a Hamiltonian cycle for $n-1$. Now we insert the number n into each permutations in all positions from right to left. For example, if $n = 4$ the sequence $\{123, 132, 312, 321, 231, 213\}$ leads to the columns of the array when 4 is inserted in all four possible positions from right to left.

$$\begin{array}{cccccc} 1234 & 1324 & 3124 & 3214 & 2314 & 2134 \\ 1243 & 1342 & 3142 & 3241 & 2341 & 2143 \\ 1423 & 1432 & 3412 & 3421 & 2431 & 2413 \\ 4123 & 4132 & 4312 & 4321 & 4231 & 4213 \end{array}$$

Now we obtain the desired sequence by reading downwards in the first column, upwards in the second, downward in the third, $\cdots$: $\{1234, 1243, 1423, 4123, 4132, 1432, \cdots, 2134, 2143, 2413, 4213\}$ The proof is simple. If we read downward or upward, just swap one pair of the adjacent numbers one of which is the number n. If we jump the next column, the position of the number n is fixed and just swap one pair of the adjacent numbers based on the induction hyperthesis.

Therefore, G_n $(n > 2)$ has a Hamiltonian cycle.

10 Special Problem 3

Let $G = (V, E)$, where $|V| = n$ and $|E| \geq \binom{n-1}{2} + 2$. Prove that G has a Hamiltonian cycle.

Answer:

Theorem 10.1 (Ore's Theorem) *A simple connected graph G on $n \geq 3$ vertices is Hamiltonian if* $\deg(u) + \deg(v) \geq n$ *for each pair of non-adjacent vertices $u, v \in V(G)$.*

Suppose G is as described. If $G = K_n$, we are done. Otherwise, there exists a pair of vertices u, v such that $(u, v) \notin E(G)$. We get a new graph G' by removing the vertices u and v and all edges involving them. There is exactly $\deg(u) + \deg(v)$ edges removed from G. So

$$E(G') = E(G) - \deg(u) - \deg(v) \geq \binom{n-1}{2} + 2 - \deg(u) - \deg(v)$$

G' has exactly $n - 2$ vertices. So $G' \subset K_{n-2}$,

$$E(G') \leq E(K_{n-2}) = \binom{n-2}{2}$$

Rearranging the inequalities, we have

$$\deg(u) + \deg(v) \geq \binom{n-1}{2} + 2 - \binom{n-2}{2} = \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2} = n$$

Applying Ore's theorem, we are done.

11 Special Problem 4

Do Exercise 3(b) in the article "The Story of Louis Pósa". Be sure that your proof is mathematically rigorous. (You should not use a computer search to solve this problem.)

Prove there is no Hamiltonian path in either of the graphs:

Answer:

Apply Lemma 4.1(b), $S = \{2, 5, 7, 9, 12, 14, 16\}$, $c(G-S) = |\{\{1\}, \{3\}, \{4\}, \{15\}, \{6\}, \{8\}, \{10\}, \{11\}, \{13\}\}| = 9 > |S| + 1 = 8$

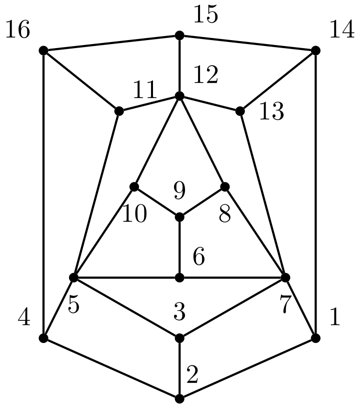

Figure 3: 3(b)