Logic for Mathematicians
Final Examination Summary

Amber
[ADN.cn]
hupo001@gmail.com

1 The System L of Propositional Logic / First-Order Predicate Logic

1.1 Propositional Logic

Axiom (Axiom 1) $\vdash_L A \to (B \to A)$

Axiom (Axiom 2) $\vdash_L (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Axiom (Axiom 3) $\vdash_L (\neg B \to \neg A) \to (A \to B)$

Assumption (Modus Ponens) $\{A, A \to B\} \vdash_L B$

1.2 First-Order Predicate Logic

Axiom (Axiom 4) $\vdash_L (\forall x)A \to A[x/t]$ where x occurs free for the term t in A.

Axiom (Axiom 5) $\vdash_L (\forall x)(A \to B) \to (A \to (\forall x)B)$ where $x \notin free(A)$

Assumption (Generalization) $\Sigma \cup \{A\} \vdash_L (\forall x)A$ where $x \notin free(\Sigma_0)$ where there exists a proof sequence from $\Sigma_0 \subseteq \Sigma$ to A.

2 The Deduction Theorem

Theorem (The Deduction Theorem) $\Sigma \underset{L}{\vdash} A \to B \iff \Sigma \cup A \underset{L}{\vdash} B$

Proof ($\Longrightarrow$) There exists a proof sequence $S_1, S_2, \ldots, S_n$ from Σ to $A \to B$. Give the proof of $\Sigma \cup A \underset{L}{\vdash} B$ as follows.

1	S_1	
2	S_2	
	$\vdots$	
n	$S_n = A \to B$	
$n+1$	A	Hypothesis
$n+2$	B	Modus Ponens $n+1$, n

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Proof ($\Longleftarrow$) Let n be the length of the proof sequence from $\Sigma \cup A$ to B.
When $n = 1$, there are 2 cases to be discussed.
Case 1: B is a axioms of L or a member of Σ. Give the proof as follows.

1	B	Axiom or Hypothesis
2	$B \to (A \to B)$	Axiom 1
3	$A \to B$	$MP12$

Case 2: B is A. Give the proof as follows.

1	$A \to ((A \to A) \to A)$	Axiom 1
2	$(A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$	Axiom 2
3	$(A \to (A \to A)) \to (A \to A)$	Modus Ponens 1, 2
4	$A \to (A \to A)$	Axiom 1
5	$A \to A$	Modus Ponens 4, 3

Suppose that for each $n < k$, the theorem is valid. When $n = k$, there are 2 cases to be discussed.
Case 1: B is A, a axioms of L or a member of Σ.
The case is the same as the case when $n = 1$.
Case 2: B is the result of Modus Ponens of C, $D = C \to B$ in the proof sequence of B. Let S be a proof sequence $S_1, S_2, \ldots, S_n$ from Σ to $A \to C$. Let T be a proof sequence $T_1, T_2, \ldots, T_m$ from Σ to $A \to D$. Give the proof of $\Sigma \underset{L}{\vdash} A \to B$ as follows.

1	S_1	
2	S_2	
	$\vdots$	
n	$S_n = A \to C$	
$n+1$	T_1	
$n+2$	T_2	
	$\vdots$	
$n+m$	$T_m = A \to D = A \to (C \to B)$	
$n+m+1$	$(A \to (C \to B)) \to ((A \to C) \to (A \to B))$	Axiom 2
$n+m+2$	$(A \to C) \to (A \to B)$	Modus Ponens $n+m$, $n+m+1$
$n+m+3$	$A \to B$	Modus Ponens n, $n+m+2$

Therefore, the theorem is valid for each natural number n. ■

Corollary (Hypothetical Syllogism) $\{A \to B, B \to C\} \underset{L}{\vdash} A \to C$

Proof $\iff \{A, A \to B, B \to C\} \underset{L}{\vdash} C$ by The Deduction Theorem

1	A	Hypothesis
2	$A \to B$	Hypothesis
3	B	Modus Ponens 1, 2
4	$B \to C$	Hypothesis
5	C	Modus Ponens 3, 4

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3 Useful Lemmas for Proposition Logic System

Lemma (Lemma 1) $\underset{L}{\vdash} \neg A \to (A \to B)$

Proof

1	$\neg A \to (\neg B \to \neg A)$	Axiom 1
2	$(\neg B \to \neg A) \to (A \to B)$	Axiom 3
3	$\neg A \to (A \to B)$	Hypothetical Syllogism 1, 2

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Lemma (Lemma 2) $\underset{L}{\vdash} (\neg A \to A) \to A$

Proof $\iff \{\neg A \to A\} \underset{L}{\vdash} A$ by The Deduction Theorem

1	$\neg A \to (A \to \neg(\neg A \to A))$	Lemma 1
2	$(\neg A \to (A \to \neg(\neg A \to A))) \to ((\neg A \to A) \to (\neg A \to \neg(\neg A \to A)))$	Axiom 2
3	$(\neg A \to A) \to (\neg A \to \neg(\neg A \to A))$	Modus Ponens 1, 2
4	$\neg A \to A$	Hypothesis
5	$\neg A \to \neg(\neg A \to A)$	Modus Ponens 4, 3
6	$(\neg A \to \neg(\neg A \to A)) \to ((\neg A \to A) \to A)$	Axiom 3
7	$(\neg A \to A) \to A$	Modus Ponens 5, 6
8	A	Modus Ponens 4, 7

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Lemma (Lemma 3) $\underset{L}{\vdash} \neg\neg A \to A$

Proof

1	$\neg\neg A \to (\neg A \to A)$	Lemma 1
2	$(\neg A \to A) \to A$	Lemma 2
3	$\neg\neg A \to A$	Hypothetical Syllogism 1, 2

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Lemma (Lemma 4) $\underset{L}{\vdash} A \to \neg\neg A$

Proof

1	$\neg\neg\neg A \to \neg A$	Lemma 3
2	$(\neg\neg\neg A \to \neg A) \to (A \to \neg\neg A)$	Axiom 3
3	$A \to \neg\neg A$	Modus Ponens 1, 2

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Lemma (Lemma 5) $\underset{L}{\vdash} (B \to A) \to (\neg A \to \neg B)$

Proof $\iff \{B \to A\} \underset{L}{\vdash} \neg A \to \neg B$ by The Deduction Theorem

1	$\neg\neg B \to B$	Lemma 3
2	$B \to A$	Hypothesis
3	$\neg\neg B \to A$	Hypothetical Syllogism 1, 2
4	$A \to \neg\neg A$	Lemma 4
5	$\neg\neg B \to \neg\neg A$	Hypothetical Syllogism 3, 4
6	$(\neg\neg B \to \neg\neg A) \to (\neg A \to \neg B)$	Axiom 3
7	$\neg A \to \neg B$	Modus Ponens 5, 6

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Lemma (Lemma 6) $\{\neg A \to B, A \to B\} \underset{L}{\vdash} B$

Proof

1	$\neg A \to B$	Hypothesis
2	$B \to \neg\neg B$	Lemma 5
3	$\neg A \to \neg\neg B$	Hypothetical Syllogism 1, 2
4	$(\neg A \to \neg\neg B) \to (\neg B \to A)$	Axiom 3
5	$\neg B \to A$	Modus Ponens 3, 4
6	$A \to B$	Hypothesis
7	$\neg B \to B$	Hypothetical Syllogism 5, 6
8	$(\neg B \to B) \to B$	Lemma 2
9	B	Modus Ponens 7, 8

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4 Applications of Lemmas in Proposition Logic

Example (Example 1) $\underset{L}{\vdash} ((A \to B) \to C) \to ((A \to C) \to C)$

Lemma (Lemma 1 for Example 1) $\underset{L}{\vdash} \neg A \to (((A \to B) \to C) \to ((A \to C) \to C))$

Proof $\iff \{\neg A, (A \to B) \to C, A \to C\} \underset{L}{\vdash} C$ by The Deduction Theorem

1	$\neg A$	Hypothesis
2	$\neg A \to (A \to B)$	Lemma 1
3	$A \to B$	Modus Ponens 1, 2
4	$(A \to B) \to C$	Hypothesis
5	C	Modus Ponens 3, 4

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Lemma (Lemma 2 for Example 1) $\underset{L}{\vdash} A \to (((A \to B) \to C) \to ((A \to C) \to C))$

Proof $\iff \{A, (A \to B) \to C, A \to C\} \underset{L}{\vdash} C$ by The Deduction Theorem

1	A	Hypothesis
2	$A \to C$	Hypothesis
3	C	Modus Ponens 1, 2

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Proof (Example 1)

1	$\neg A \to (((A \to B) \to C) \to ((A \to C) \to C))$	Lemma 1 for Example 1
2	$A \to (((A \to B) \to C) \to ((A \to C) \to C))$	Lemma 2 for Example 1
3	$((A \to B) \to C) \to ((A \to C) \to C)$	Lemma 6

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5 Useful Lemmas for First-Order Predicate Logic

6 Applications of Lemmas in First-Order Predicate Logic